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Double extension regular algebras of type (14641)

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A R T I C L E I N F O

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A B S T R A C T

We construct several families of Artin–Schelter regular algebras of global dimension four using double Ore extension and then prove that all these algebras are strongly noetherian, Auslander regular, Koszul and Cohen–Macaulay domains. Many regular algebras constructed in the paper are new and are not isomorphic to either a normal extension or an Ore extension of an Artin–Schelter regular algebra of global dimension three.

Introduction

One of the most important projects in non-commutative algebraic geometry is the classification of non-commutative projective 3-spaces or quantum $\mathbb{P}^3$s. An algebraic version of this project is the classification of Artin–Schelter regular algebras of global dimension four. There has been extensive research on Artin–Schelter regular algebras of global dimension four; and many families of regular algebras have been discovered in recent years [8,10,15–17,19–23]. The main goal of this paper is to construct and study a large class of new Artin–Schelter regular algebras of dimension four, called double Ore extensions.

The notion of double Ore extension (or double extension for the rest of the paper) was introduced in [28]. A double extension of an algebra A is denoted by $A_P[y_1, y_2; \sigma, \delta, \tau]$ and the meanings of the DE-data $\{P, \sigma, \delta, \tau\}$ will be reviewed in Section 1. A more general way of building regular algebras of dimension four was presented by Caines in his thesis [6]. In principle, all double extensions in this paper are also "skew-polynomial rings" in the sense of Caines. The idea of double extensions was used by Patrick [14] and Nyman [12] in a different context. As a generalization of the classical

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Ore extension [13] the method of double Ore extension is simple and effective. By using the double extension we construct regular algebras of global dimension four explicitly.

Many researchers have noted that the data associated to the whole class of regular algebras of dimension four are tremendous; and one needs to introduce some invariants to distinguish these algebras. For simplicity we only consider regular algebras of dimension four that are generated in degree 1. By the work of [10], such an algebra B is generated by either 2, or 3, or 4 elements and the projective resolution of the trivial module k_B is given in [10, Proposition 1.4]. When B is generated by 4 elements, then the projective resolution of the trivial module k_B is of the form

$$0 \to B(-4) \to B(-3)^{\oplus 4} \to B(-2)^{\oplus 6} \to B(-1)^{\oplus 4} \to B \to k_B \to 0.$$

Suggested by the form of the above resolution, we say such an algebra is *of type* (14641). In this paper we mainly deal with algebras of type (14641). An algebra of type (14641) is Koszul.

With some help of the Mathematical software Maple, we are able to classify all double extensions $A_P[y_1, y_2; \sigma]$ (with $\delta = 0$ and $\tau = (0, 0, 0)$) of type (14641). Since Ore extensions and normal extensions of regular algebras of dimension three are well understood and well studied [8], we omit some of those from our classification. Our then "partial" classification consists of 26 families of regular algebras of type (14641); and it provides enough information to prove part (a) of the following theorem.

Theorem 0.1. *Let B be a connected graded algebra generated by four elements of degree* 1. *Suppose that B is a double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ where A is an Artin–Schelter regular algebra of dimension* 2.

(a) *B is a strongly noetherian, Auslander regular and Cohen–Macaulay domain.*
(b) *B is of type* (14641). *As a consequence, B is Koszul.*
(c) *If B is not isomorphic to an Ore extension of an Artin–Schelter regular algebra of dimension three, then the trimmed double extension $A_P[y_1, y_2; \sigma]$ (by setting $\delta = 0$ and $\tau = (0, 0, 0)$) is isomorphic to one of* 26 *families listed in Section* 4.

These 26 families of algebras in part (c) are labeled by $\mathbb{A}, \mathbb{B}, \ldots, \mathbb{Z}$. Let $\mathcal{LIST}$ denote the class consisting of all algebras in the families from $\mathbb{A}$ to $\mathbb{Z}$. Regular algebras of dimension three are well understood [1,3,4]. Hence, in theory, Ore extensions of regular algebras of dimension three are well understood. That is our rationale to omit Ore extensions in part (c). Besides, there are too many double extensions $A_P[y_1, y_2; \sigma, \delta, \tau]$ to list if we want to include all Ore extensions. On the other hand, in these 26 families, many of the double extensions $A_P[y_1, y_2; \sigma]$ are still Ore extensions. The reason that we do not remove those Ore extensions is that there might be non-zero δ and τ such that $A_P[y_1, y_2; \sigma, \delta, \tau]$ (with the same (P, σ)) is not an Ore extension. In other words our classification is basically the classification of (P, σ) so that $A_P[y_1, y_2; \sigma, \delta, \tau]$ is not an Ore extension for possible (δ, τ).

We want to remark that the software Maple is used in an elementary way only to reduce the length of the computation and all computation can be done by hand without assistance of Maple. Further, the regularity and other properties of every algebra are verified rigorously by other means.

As a consequence of Theorem 0.1, for any algebra A in the $\mathcal{LIST}$, the scheme of point modules (respectively, line modules) over A is a genuine commutative projective scheme [5, Corollary E4.11]. It would be very interesting to work out geometric properties and geometric invariants (such as the point-scheme and the line-scheme) associated to A. There are also various algebraic questions we do not pursue in this paper. For example, for any algebra A in the $\mathcal{LIST}$, one may ask:

(a) Is A primitive? Does A satisfy a polynomial identity? What is the prime spectrum Spec A?
(b) What is the group of graded algebra automorphisms of A (denoted by $\mathrm{Aut}(A)$)? Is there a non-trivial finite subgroup $G \subset \mathrm{Aut}(A)$ such that A^G is Artin–Schelter regular?
(c) What invariants can be defined for the quotient division algebra of A? Is the quotient division algebra of A always generated by two elements?

Some of these questions are easy for each individual algebra; however, it could be a challenge to find a general approach that works for all algebras. Question (b) leads to finding more regular algebras of dimension four that may not be generated in degree 1.

Double extensions appear naturally in some slightly different contexts about regular algebras of type (14641).

Theorem 0.2. *Let B be a noetherian Artin–Schelter regular algebra of type* (14641)*. Suppose that B is $\mathbb{Z}^2$-graded with a decomposition $B_1 = B_{01} \oplus B_{10}$ where B_{01} and B_{10} are non-zero $\mathbb{Z}^2$-homogeneous components.*

(a) *If $\dim B_{01} = 1$ or $\dim B_{10} = 1$, then B is isomorphic to an Ore extension $A[y;\sigma]$ for some Artin–Schelter regular algebra A of dimension three.*
(b) *If $\dim B_{01} = \dim B_{10} = 2$, then B is isomorphic to a trimmed double extension $A_P[y_1, y_2;\sigma]$ for some Artin–Schelter regular algebra A of dimension two.*

Since all double extensions $A_P[y_1, y_2;\sigma]$ (that are not Ore extensions of regular algebras of dimensional three) are classified in Section 4, Theorem 0.2 gives a classification of (non-trivially) $\mathbb{Z}^2$-graded noetherian regular algebras of type (14641). Various other properties related to Artin–Schelter regular algebras are studied. Here is another characterization of the double extensions in Theorem 0.1.

Proposition 0.3. *Let B be an Artin–Schelter regular domain of global dimension four generated by four degree* 1 *elements. Then B is a double extension if and only if there are $x_1, x_2 \in B_1$ such that*

(a) *B has a quadratic relation involving only x_1, x_2,*
(b) *$B/(x_1, x_2)$ is Artin–Schelter regular of dimension two.*

Here is an outline of the paper. In Section 1 we review some basic definitions. Theorem 0.2 and Proposition 0.3 are proved in Section 2. Sections 3 and 4 are devoted to the classification that is unfortunately very tedious. Our main Theorem 0.1 is proved in Section 5.

1. Definitions

Throughout k is a commutative base field, that is algebraically closed. Everything is over k; in particular, an algebra or a ring is a k-algebra. An algebra A is called *connected graded* if

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

with $1 \in k = A_0$ and $A_i A_j \subset A_{i+j}$ for all i, j. If A is connected graded, then k also denotes the trivial graded module $A/A_{\geqslant 1}$. In this paper we are working on connected graded algebras. One basic concept we will use is Artin–Schelter regularity, which we now review. A connected graded algebra A is called *Artin–Schelter regular* or *regular* for short if the following three conditions hold.

(AS1) A has finite global dimension d.
(AS2) A is *Gorenstein*, namely, there is an integer l such that,

$$\operatorname{Ext}^i_A({}_A k, A) = \begin{cases} k(l) & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

where k is the trivial A-module; and the same condition holds for the right trivial A-module k_A.
(AS3) A has finite Gelfand–Kirillov dimension, i.e., there is a positive number c such that $\dim A_n < cn^c$ for all $n \in \mathbb{N}$.

If A is regular, then the global dimension of A is called the *dimension* of A. The notation (l) in (AS2) is the lth degree shift of graded modules.

Definition 1.1. Let A, B and C be connected graded algebras. The algebra B is called an *extension* of $(A|C)$, if there is a sequence of graded maps

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

satisfying the following conditions:

(a) Both f and g are graded algebra homomorphisms.
(b) B contains A as a graded subalgebra via f.
(c) $A_{\geqslant 1}B = BA_{\geqslant 1}$ and the map g induces an isomorphism of graded algebras $B/(A_{\geqslant 1}) \cong C$.
(d) There is a vector space $\bar{C} \subset B$ such that $g : \bar{C} \to C$ is an isomorphism of graded vector spaces and B is a left and a right free A-module with basis $\bar{C}$.

If C is a regular algebra of dimension n, we call B an n-extension of A. If A is a regular algebra of dimension n, then we call B an n-co-extension of C.

The following lemma characterizes n-(co)-extensions for $n = 0, 1$.

Lemma 1.2.

(a) *B is a* 0*-extension of A if and only if $f : A \to B$ is an isomorphism.*
(b) *B is a* 0*-co-extension of C if and only if $g : B \to C$ is an isomorphism.*
(c) *Suppose A is generated in degree* 1*. Then B is a* 1*-extension of A if and only if B is an Ore extension of A.*
(d) *B is a* 1*-co-extension of C if and only if B is a normal extension of C, namely, there is a normal regular element t such that $B/(t) \cong C$.*

Proof. (a), (b) Trivial.

(c) If B is an Ore extension $A[x; \sigma, \delta]$, then it is easy to check that B is a 1-extension of A (without assuming that A is generated in degree 1).

Now we assume B is a 1-extension of A and A is generated by A_1. In this case $C = k[x]$. Let $\bar{C} = \bigoplus_{i \geqslant 0} kx_i$ such that $g(x_i) = x^i$. For every $a \in A_1$, $ax_1 \in B = A \oplus \bigoplus_{i \geqslant 1} x_i A$ and we can write

$$ax_1 = b_0 + \sum_{i \geqslant 1} x_i b_i$$

with $b_0, b_1, \ldots \in A$. Since $\deg(ax_1) = 1 + \deg x_1 \leqslant \deg x_2 < \deg x_i$ for all $i > 2$, $b_i = 0$ for all $i > 2$ and $b_2 \in k$. Since $A_{\geqslant 1}B = BA_{\geqslant 1}$, $b_2 = 0$. Thus $ax_1 \in A \oplus x_1 A$. This implies that $Ax_1 \subset A \oplus x_1 A$. By symmetry, $x_1 A \subset A \oplus Ax_1$. So there are maps σ and δ such that

$$x_1 r = \sigma(r)x_1 + \delta(r)$$

for all $r \in A$. It is easy to see that σ is an algebra automorphism of A and δ is a σ-derivation of A. Finally it is routine to check that $B = A[x_1; \sigma, \delta]$.

(d) In this case $A = k[t]$. Since B is a free A-module on both sides, t is regular on both sides. By part (c) of Definition 1.1, $A_{\geqslant 1}B = BA_{\geqslant 1}$. This implies that $tB = Bt = (A_{\geqslant 1})$. So t is a regular normal element of B and $B/(t) = C$. This says that B is a normal extension of C. The converse is clear. $\square$

If A is not generated in degree 1, then it is possible that a 1-extension is not an Ore extension. Most algebras in this paper will be generated in degree 1. Some basic properties of Ore extensions can be found in [11, Chapter 1].

Next we will show that the definition of a 2-extension is equivalent to that of a double extension introduced in [28]. We first review the definition of a double extension in the connected graded case.

Definition 1.3. (See [28, Definition 1.3].) Let A be a connected graded algebra and B be another connected graded algebra containing A as a graded subring.

(a) We say B is a *right double extension* of A if the following conditions hold.
  (ai) B is generated by A and two variables y_1 and y_2 of positive degree.
  (aii) $\{y_1, y_2\}$ satisfies a homogeneous relation

$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0 \tag{R1}$$

  where $p_{12}, p_{11} \in k$ and $\tau_1, \tau_2, \tau_0 \in A$.
  (aiii) As a left A-module, $B = \sum_{n_1, n_2 \geqslant 0} A y_1^{n_1} y_2^{n_2}$ and it is a left free A-module with a basis $\{y_1^{n_1} y_2^{n_2} \mid n_1 \geqslant 0,\ n_2 \geqslant 0\}$.
  (aiv) $y_1 A + y_2 A \subseteq A y_1 + A y_2 + A$.

  Let P denote the set of scalar parameters $\{p_{12}, p_{11}\}$ and let τ denote the set $\{\tau_1, \tau_2, \tau_0\}$.

(b) We say B is a *left double extension* of A if the following conditions hold.
  (bi) B is generated by A and two variables y_1 and y_2.
  (bii) $\{y_1, y_2\}$ satisfies a homogeneous relation

$$y_1 y_2 = p'_{12} y_2 y_1 + p'_{11} y_1^2 + y_1 \tau'_1 + y_2 \tau'_2 + \tau'_0 \tag{L1}$$

  where $p'_{12}, p'_{11} \in k$ and $\tau'_1, \tau'_2, \tau'_0 \in A$.
  (biii) As a right A-module, $B = \sum_{n_1, n_2 \geqslant 0} y_2^{n_1} y_1^{n_2} A$ and it is a right free A-module with a basis $\{y_2^{n_1} y_1^{n_2} \mid n_1 \geqslant 0,\ n_2 \geqslant 0\}$.
  (biv) $A y_1 + A y_2 A \subseteq y_1 A + y_2 A + A$.

(c) We say B is a *double extension* if it is a left and a right double extension of A with the same generating set $\{y_1, y_2\}$.

If B is a double extension of A, then $p_{12} p'_{12} = 1$ and hence $p_{12} \neq 0$. Both Definitions 1.1 and 1.3 are abstract. To study extensions we need to find more precise information about these algebras. The condition in Definition 1.3(aiv) can be written as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} r := \begin{pmatrix} y_1 r \\ y_2 r \end{pmatrix} = \begin{pmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \delta_1(r) \\ \delta_2(r) \end{pmatrix} \tag{R2}$$

for all $r \in A$. Here $\sigma(r) := \left(\begin{smallmatrix} \sigma_{11}(r) & \sigma_{12}(r) \\ \sigma_{21}(r) & \sigma_{22}(r) \end{smallmatrix}\right)$ is an algebra homomorphism from A to $M_2(A)$ and $\delta(r) := \left(\begin{smallmatrix} \delta_1(r) \\ \delta_2(r) \end{smallmatrix}\right)$ is a σ-derivation from A to $A^{\oplus 2} := \left(\begin{smallmatrix} A \\ A \end{smallmatrix}\right)$. By [28, Section 1], σ and δ are uniquely determined. Together with Definition 1.3, all symbols in the DE-data $\{P, \sigma, \delta, \tau\}$ are defined, and the double extension B in Definition 1.3 is denoted by $A_P[y_1, y_2; \sigma, \delta, \tau]$. We call σ a *homomorphism*, δ a *derivation*, P a *parameter* and τ a *tail*. A double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ is called *trimmed* if $\delta = 0$ and $\tau = \{0, 0, 0\}$.

Lemma 1.4. *Suppose A and B are generated in degree* 1*. Then B is a* 2*-extension of A if and only if B is a double extension of A.*

Proof. If B is a double extension of A, then by [28, Proposition 1.14], $A_{\geqslant 1} B = B A_{\geqslant 1}$, which is denoted by $(A_{\geqslant 1})$, and

$$B/(A_{\geqslant 1}) = k\langle Y_1, Y_2\rangle / \bigl(Y_2 Y_1 - p_{12} Y_1 Y_2 - p_{11} Y_1^2\bigr)$$

where Y_i is the image of y_i in $B/(A_{\geqslant 1})$. So $C = k\langle Y_1, Y_2\rangle / (Y_2 Y_1 - p_{12} Y_1 Y_2 - p_{11} Y_1^2)$ and it is regular of dimension 2. Condition (d) in the definition of 2-extension follows from Definition 1.3(aiii), (biii).

Conversely, we assume that B is a 2-extension. So there is an "exact sequence"

$$0 \to A \to B \to C \to 0$$

where C is regular of dimension 2. Since C is generated in degree 1, C is isomorphic to $k\langle Y_1, Y_2\rangle/(Y_2Y_1 - p_{12}Y_1Y_2 - p_{11}Y_1^2)$ for some $p_{12}, p_{11} \in k$ and $p_{12} \neq 0$ [27, Theorem 0.2]. Lift Y_1 and Y_2 to y_1 and y_2 in B. Then y_1 and y_2 satisfy

$$y_2y_1 - p_{12}y_1y_2 - p_{11}y_1^2 \in A_{\geqslant 1}B,$$

since $A_{\geqslant 1}B$ is the kernel of the homomorphism $B \to C$. This implies that

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0 \tag{E1.4.1}$$

for some $\tau_0 \in A_2$ and $\tau_1, \tau_2 \in A_1$. This gives (R1) and therefore Definition 1.3(aii) holds.

Since $A_{\geqslant 1}B = BA_{\geqslant 1}$ [Definition 1.1(c)],

$$A_1y_1 + A_1y_2 \subset (A_{\geqslant 1}B)_2 = (BA_{\geqslant 1})_2 = y_1A_1 + y_2A_1 + A_2.$$

Since A is generated by A_1, by induction on the degree of elements in A, we obtain that

$$Ay_1 + Ay_2 \subset y_1A + y_2A + A.$$

Similarly, $y_1A + y_2A \subset Ay_1 + Ay_2 + A$. Consequently,

$$Ay_1 + Ay_2 + A = y_1A + y_2A + A \tag{E1.4.2}$$

and it is a free A-module of rank 3 by Definition 1.1(d). Definition 1.3(ai) is clear and Definition 1.3(aiv) is (E1.4.2).

Next we show Definition 1.3(aiii). By using (E1.4.1) and (E1.4.2), every element in B can be written as $\sum_{n_1,n_2 \geqslant 0} a_{n_1,n_2} y_1^{n_1} y_2^{n_2}$ for $a_{n_1,n_2} \in A$. This implies that $B = \sum_{n_1,n_2 \geqslant 0} Ay_1^{n_1}y_2^{n_2}$. Since B is a 2-extension, by Definition 1.1(d), the Hilbert series of B is $H_B(t) = H_A(t)H_C(t)$. Thus B is a free left A-module with basis $\{y_1^{n_1}y_2^{n_2} \mid n_1, n_2 \geqslant 0\}$. So Definition 1.3(aiii) holds, and B is a right double extension of A. By symmetry, B is a left double extension of A with the same generating set $\{y_1, y_2\}$. Therefore B is a double extension. □

The following definition is given in [28, Definition 1.8].

Definition 1.5. (See [28, Definition 1.8].) Let $\sigma : A \to M_2(A)$ be an algebra homomorphism. We say σ is *invertible* if there is an algebra homomorphism

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \to M_2(A)$$

which satisfies the following conditions:

$$\sum_{k=1}^{2} \phi_{jk}\bigl(\sigma_{ik}(r)\bigr) = \begin{cases} r & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{and} \quad \sum_{k=1}^{2} \sigma_{kj}\bigl(\phi_{ki}(r)\bigr) = \begin{cases} r & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $r \in A$, or equivalently,

$$\begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \bullet \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \bullet \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} Id_A & 0 \\ 0 & Id_A \end{pmatrix}$$

where $\bullet$ is the multiplication of the matrix algebra $M_2(\mathrm{End}_k(A))$. The multiplication of $\mathrm{End}_k(A)$ is the composition of k-linear maps. The map ϕ is called the *inverse* of σ.

By [28, Lemma 1.9] if $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ is a double extension of A, then σ is invertible in the sense of Definition 1.5. As in [28, Section 3], one can define the determinant of σ, denoted by $\det \sigma$. By [28, Section 4] $\det \sigma$ plays an essential role in the proof of regularity of double extensions.

Next we will list the relations (or the constraints) between the DE-data that come from commuting $r \in A$ with (R1). The collection of the following six relations is called (R3) for short.

Relations (R3)

$$\begin{aligned}
&\sigma_{21}(\sigma_{11}(r)) + p_{11}\sigma_{22}(\sigma_{11}(r)) \\
&\quad = p_{11}\sigma_{11}(\sigma_{11}(r)) + p_{11}^2\sigma_{12}(\sigma_{11}(r)) + p_{12}\sigma_{11}(\sigma_{21}(r)) + p_{11}p_{12}\sigma_{12}(\sigma_{21}(r)), \qquad \text{(R3.1)}
\end{aligned}$$

$$\begin{aligned}
&\sigma_{21}(\sigma_{12}(r)) + p_{12}\sigma_{22}(\sigma_{11}(r)) \\
&\quad = p_{11}\sigma_{11}(\sigma_{12}(r)) + p_{11}p_{12}\sigma_{12}(\sigma_{11}(r)) + p_{12}\sigma_{11}(\sigma_{22}(r)) + p_{12}^2\sigma_{12}(\sigma_{21}(r)), \qquad \text{(R3.2)}
\end{aligned}$$

$$\sigma_{22}(\sigma_{12}(r)) = p_{11}\sigma_{12}(\sigma_{12}(r)) + p_{12}\sigma_{12}(\sigma_{22}(r)), \qquad \text{(R3.3)}$$

$$\begin{aligned}
&\sigma_{20}(\sigma_{11}(r)) + \sigma_{21}(\sigma_{10}(r)) + \tau_1\sigma_{22}(\sigma_{11}(r)) \\
&\quad = p_{11}\big[\sigma_{10}(\sigma_{11}(r)) + \sigma_{11}(\sigma_{10}(r)) + \tau_1\sigma_{12}(\sigma_{11}(r))\big] \\
&\qquad + p_{12}\big[\sigma_{10}(\sigma_{21}(t)) + \sigma_{11}(\sigma_{20}(r)) + \tau_1\sigma_{12}(\sigma_{21}(r))\big] + \tau_1\sigma_{11}(r) + \tau_2\sigma_{21}(r), \qquad \text{(R3.4)}
\end{aligned}$$

$$\begin{aligned}
&\sigma_{20}(\sigma_{12}(r)) + \sigma_{22}(\sigma_{10}(r)) + \tau_2\sigma_{22}(\sigma_{11}(r)) \\
&\quad = p_{11}\big[\sigma_{10}(\sigma_{12}(r)) + \sigma_{12}(\sigma_{10}(r)) + \tau_2\sigma_{12}(\sigma_{11}(r))\big] \\
&\qquad + p_{12}\big[\sigma_{10}(\sigma_{22}(r)) + \sigma_{12}(\sigma_{20}(r)) + \tau_2\sigma_{12}(\sigma_{21}(r))\big] + \tau_1\sigma_{12}(r) + \tau_2\sigma_{22}(r), \qquad \text{(R3.5)}
\end{aligned}$$

$$\begin{aligned}
&\sigma_{20}(\sigma_{10}(r)) + \tau_0\sigma_{22}(\sigma_{11}(r)) \\
&\quad = p_{11}\big[\sigma_{10}(\sigma_{10}(r)) + \tau_0\sigma_{12}(\sigma_{11}(r))\big] \\
&\qquad + p_{12}\big[\sigma_{10}(\sigma_{20}(r)) + \tau_0\sigma_{12}(\sigma_{21}(r))\big] + \tau_1\sigma_{10}(r) + \tau_2\sigma_{20}(r) + \tau_0 r. \qquad \text{(R3.6)}
\end{aligned}$$

The following is a combination of [28, Propositions 1.11 and 1.13].

Proposition 1.6. *Let A be an algebra. Let $\{P, \sigma, \delta, \tau\}$ be a set of data such that $\sigma : A \to M_2(A)$ is an algebra homomorphism and $\delta : A \to A^{\oplus 2}$ is a σ-derivation and that $P = \{p_{12}, p_{11}\} \subseteq k$ and $\tau = \{\tau_1, \tau_2, \tau_0\} \subseteq A$.*

(a) *Assume that* (R3) *holds for all $r \in X$ where X is a set of generators of A. Let B be the algebra generated by A and y_1, y_2 subject to the relations* (R1) *and* (R2) *for generators $r \in X$. Then B is a right double extension of A. Namely, B is a left free A-module with a basis $\{y_1^{n_1} y_2^{n_2} \mid n_1, n_2 \geqslant 0\}$.*
(b) *If further B is connected graded, $p_{12} \neq 0$ and σ is invertible, then B is a double extension of A.*

2. Regular algebras of dimension four

In this section we discuss some homological properties of (Artin–Schelter) regular algebras. We assume that all graded algebras in this section are generated in degree 1.

The definition of regularity is recalled in Section 1. If B is regular, then by [18, Proposition 3.1.1], the trivial left B-module ${}_Bk$ has a minimal free resolution of the form

$$0 \to P_d \to \cdots \to P_1 \to P_0 \to k_B \to 0 \tag{E2.0.1}$$

where $P_w = \bigoplus_{s=1}^{n_w} B(-i_{w,s})$ for some finite integers n_w and $i_{w,s}$. The Gorenstein condition (AS2) implies that the above free resolution is symmetric in the sense that the dual complex of (E2.0.1) is a free resolution of the trivial right B-module (after a degree shift). As a consequence, we have $P_0 = B$, $P_d = B(-l)$, $n_w = n_{d-w}$, and $i_{w,s} + i_{d-w,n_w-s+1} = l$ for all w, s.

Regular algebras of dimension three have been classified by Artin, Schelter, Tate and Van den Bergh [1,3,4]. If B is a regular algebra of dimension three, then it is generated by either two or three elements. If B is generated by three elements, then B is Koszul and the trivial B-module k has a minimal free resolution of form

$$0 \to B(-3) \to B(-2)^{\oplus 3} \to B(-1)^{\oplus 3} \to B \to k \to 0.$$

If B is generated by two elements, then B is not Koszul and the trivial B-module k has a minimal free resolution of the form

$$0 \to B(-4) \to B(-3)^{\oplus 2} \to B(-1)^{\oplus 2} \to B \to k \to 0.$$

If B is a noetherian regular algebra of (global) dimension four, then B is generated by either 2, or 3, or 4 elements [10, Proposition 1.4]. Minimal free resolutions of the trivial module k are listed in [10, Proposition 1.4]. The following lemma is well known. The transpose of a matrix M is denoted by M^T.

Lemma 2.1. *Let B be a regular domain of dimension four. Suppose B is generated by elements x_1, x_2, x_3, x_4 (of degree 1).*

(a) *B is of type (14641), namely, the trivial left B-module k has a free resolution*

$$0 \to B(-4) \xrightarrow{\partial_4} B^{\oplus 4}(-3) \xrightarrow{\partial_3} B^{\oplus 6}(-2) \xrightarrow{\partial_2} B^{\oplus 4}(-1) \xrightarrow{\partial_1} B \xrightarrow{\partial_0} k \to 0 \tag{E2.1.1}$$

where $B^{\oplus n}$ is the free left B-module written as a $1 \times n$ matrix.

(b) *∂_0 is the augmentation map with $\ker \partial_0 = B_{\geqslant 1}$.*

(c) *∂_1 is given by the right multiplication by $(x_1, x_2, x_3, x_4)^T$.*

(d) *∂_2 is the right multiplication by a 6×4-matrix $F = (f_{ij})_{6\times 4}$ such that $f_i := \sum_{j=1}^4 f_{ij}x_j$, for $i = 1, 2, 3, 4, 5, 6$, are the 6 relations of B.*

(e) *∂_3 is the right multiplication by a 4×6-matrix $G = (g_{ij})_{4\times 6}$.*

(f) *∂_4 is the right multiplication by (x'_1, x'_2, x'_3, x'_4) where $\{x'_1, x'_2, x'_3, x'_4\}$ is a set of generators of B. (So each x'_i is a k-linear combination of $\{x_i\}_{i=1}^4$.)*

(g) *$F(x_1, x_2, x_3, x_4)^T = 0$, $GF = 0$, $(x'_1, x'_2, x'_3, x'_4)G = 0$.*

The dual complex of (E2.1.1) is obtained by applying the functor $(-)^\vee := \mathrm{Hom}_B(-, B)$ to (E2.1.1). Condition (AS2) implies that the dual complex of (E2.1.1) is a free resolution of the right B-module $k(4)$:

$$0 \leftarrow k_B(4) \leftarrow B(4) \xleftarrow{\partial_4^\vee} B^{\oplus 4}(3) \xleftarrow{\partial_3^\vee} B^{\oplus 6}(2) \xleftarrow{\partial_2^\vee} B^{\oplus 4}(1) \xleftarrow{\partial_1^\vee} B \leftarrow 0. \tag{E2.1.2}$$

Lemma 2.1(f) follows from this observation. The other parts of Lemma 2.1 are clear.

Lemma 2.2. *Let B be as in Lemma* 2.1.

(a) *Each column and each row of F and G is non-zero.*
(b) *If α is a non-zero row vector in k^4 and β is a non-zero row vector in k^6, then $F\alpha^T \neq 0$, $\alpha G \neq 0$, $\beta F \neq 0$ and $G\beta^T \neq 0$.*
(c) *The subspace spanned by elements in a fixed column* (*or row*) *of either F or G has dimension at least* 2.

Proof. (a) If a row of F is zero, then $\ker \partial_2$ contains a copy of $B(-2)$ and (E2.1.1) is not exact; that is a contradiction. So any row of F is non-zero. Suppose now a column of F is zero. Consider the complex (E2.1.2) where each $B^{\oplus n}$ is an n-column right free B-module. This complex is a free resolution of $k_B(4)$ by the Gorenstein condition (AS2). The map $\partial_2^\vee$ is left multiplication by F. If some column of F is zero, then $\ker \partial_2^\vee$ contains a copy of $B(1)$. So complex (E2.1.2) is not exact at $B^{\oplus 4}(1)$, a contradiction.

The same proof works for G.

(b) Let M be a 4×4 non-singular matrix such that α^T is the first column of M^{-1}. Replacing $X := \{x_1, x_2, x_3, x_4\}^T$ by another generating set $X' := MX$ will change F to $F' := FM^{-1}$. The first column of F' is zero if $F\alpha^T = 0$. This contradicts part (a). So $F\alpha^T \neq 0$. Similarly, $\alpha G \neq 0$ by passing to the dual complex and applying the proof of the first case.

For $\beta F \neq 0$, we use a change of basis of $B^{\oplus 6}$ in the middle of (E2.1.1). The proof is similar to the last paragraph and is omitted. One can repeat the proof of $\beta F \neq 0$ for $G\beta^T \neq 0$ by considering the dual complex (E2.1.2).

(c) If the dimension of the subspace spanned by the first column of F is 1 (which cannot be zero by part (b)), then $f_{i1} = \beta_i v$ for some $\beta_i \in k$ and $0 \neq v \in B_1$. Then $G\beta^T v = 0$ for $\beta = (\beta_1, \ldots, \beta_6)$. Since B is a domain, we have $G\beta^T = 0$, which contradicts with part (b).

If the dimension of the subspace spanned by the first column of G is 1, then $g_{i1} = \alpha_i v$ for some $\alpha_i \in k$ and $0 \neq v \in B_1$. Then $(x'_1, x'_2, x'_3, x'_4)\alpha^T v = 0$. Since B is a domain $(x'_1, x'_2, x'_3, x'_4)\alpha^T = 0$. This implies that x'_1, x'_2, x'_3, x'_4 are k-linearly dependent, a contradiction.

By using (E2.1.2) we can prove the assertion for the rows of F and G. □

Lemma 2.2 can be used to show some graded algebras are not regular. Here is an example.

Proposition 2.3. *Let B be a graded domain generated by elements x_1, x_2, x_3, x_4. Suppose B has six quadratic relations of following form*:

$$x_1 x_4 = q x_4 x_1, \quad \textit{where } q \in k;$$

$$x_4^2 = f(x_1, x_2, x_3); \quad \textit{and}$$

$$4 \textit{ other relations only involving } x_1, x_2, x_3.$$

Assume that $x_4^2 \neq g x_1$ for any $g \in kx_1 + kx_2 + kx_3$. Then B is not regular of dimension four.

Proof. Suppose on the contrary that B is regular of dimension four. Then we can use Lemma 2.2 and use the notations introduced there. The form of the relations implies that

$$F = \begin{pmatrix} -qx_4 & 0 & 0 & x_1 \\ * & * & * & x_4 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{pmatrix}$$

where all $*$ are in $kx_1 + kx_2 + kx_3$. Since $GF = 0$, the 4th column of this matrix equation gives $g_{i1}x_1 + g_{i2}x_4 = 0$ for all $i = 1, 2, 3, 4$. Looking back at the 6 relations of B, one sees that only the first two relations have the term gx_4. This implies that $g_{i2} = a_i x_1 + b_i x_4$ for some $a_i, b_i \in k$. But by the first two relations

$$(a_i x_1 + b_i x_4)x_4 = a_i q x_4 x_1 + b_i f(x_1, x_2, x_3)$$

where the right-hand side is not of the form $g' x_1$ unless $b_i = 0$ for all i, since we assume that $x_4^2 \neq g x_1$. Therefore $g_{i1}x_1 + g_{i2}x_4 = 0$ implies that $b_i = 0$ for all i. Now the space spanned by the second column of G is kx_1, which is 1-dimensional. We obtain a contradiction by Lemma 2.2(c). Therefore B is not regular of dimension four. $\square$

Next we want to show that if a right double extension is a regular domain of type (14641), then it is automatic a double extension.

Lemma 2.4. *Let A be a connected graded algebra and let $B = A_P[y_1, y_2; \sigma, \delta, \tau]$ be a right double extension of A. Suppose B is a regular domain of type* (14641)*. Then A is a regular domain of dimension* 2 *with Hilbert series $(1-t)^{-2}$, namely, A is isomorphic to either $k_p[x_1, x_2]$ or $k_J[x_1, x_2]$.*

Proof. Since $H_B(t) = (1-t)^{-4}$ and B is a free A-module with basis $\{y_1^{n_1} y_2^{n_2}\}_{n_1, n_2 \geqslant 0}$, we see that $H_A(t) = (1-t)^{-2}$.

Let A' be the subalgebra of A generated by the elements of degree 1. Then A' is a domain and generated by two elements of degree 1, say x_1, x_2. Since $H_A(t) = (1-t)^{-2}$, A (and hence A') has at least one relation in degree 2. Any graded domain generated by two elements with at least one relation in degree 2 is isomorphic to $k\langle x_1, x_2\rangle/(x_2x_1 - p_{12}x_1x_2 - p_{11}x_1^2)$ for some $p_{ij} \in k$ with $p_{12} \neq 0$ (this is well known and a proof is given in [7, Lemma 3.7]). Therefore $A' = k\langle x_1, x_2\rangle/(x_2x_1 - p_{12}x_1x_2 - p_{11}x_1^2)$ and hence $H_{A'}(t) = (1-t)^{-2}$. Thus A and A' have the same Hilbert series, whence $A = A'$. In particular, A (and hence B) has a relation

$$x_2 x_1 = p_{12} x_1 x_2 + p_{11} x_1^2.$$

After a linear transformation (p_{12}, p_{11}) can be chosen to be either $(p, 0)$ or $(1, 1)$ that corresponds to regular algebras $k_p[x_1, x_2]$ and $k_J[x_1, x_2]$ respectively. $\square$

Let V denote the vector space $kx_1 + kx_2$ and W denote the vector space $V + ky_1 + ky_2$. By Lemma 2.4, B contains a relation in $V \otimes V$. Recall that a right double extension B has a relation (R1):

$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + \tau_0.$$

The condition (R2) implies that there are 4 relations in $V \otimes W + W \otimes V$.

Lemma 2.5. *Let B be a regular domain of type* (14641)*. Suppose that B is generated by x_1, x_2, y_1, y_2 satisfying the following quadratic relations:*

(i) *$f_1 = 0$ for some $f_1 \in V \otimes V$.*
(ii) *Four relations $f_i = 0$ for $i = 2, 3, 4, 5$ where $f_i \in V \otimes W + W \otimes V$.*
(iii) *One relation f_6 of the modified form of* (R1)

$$y_2 y_1 = p_{12} y_1 y_2 + p_{11} y_1^2 + \tau_1 y_1 + \tau_2 y_2 + y_1 \tau_1' + y_2 \tau_2' + \tau_0 \qquad \text{(E2.5.1)}$$

with $p_{12}, p_{11} \in k$ and $0 \neq p_{12}$, and where $\tau_1, \tau_2, \tau_1', \tau_2' \in V$ and $\tau_0 \in V \otimes V$.

Then

(a) $VW = WV$ *in* B.
(b) *Let* A *be the subalgebra of* B *generated by* V. *Then* $Ay_1 + Ay_2 + A = y_1A + y_2A + A$.
(c) B *is a double extension* $A_P[y_1, y_2; \sigma, \delta, \tau]$. *In particular,* σ *is invertible.*

Proof. (a) The assertion is equivalent to equation

$$Vy_1 + Vy_2 + V^2 = y_1V + y_2V + V^2.$$

Suppose this is not true. Then we have the following two cases: either

$$Vy_1 + Vy_2 + V^2 \subsetneq y_1V + y_2V + V^2$$

or

$$Vy_1 + Vy_2 + V^2 \supsetneq y_1V + y_2V + V^2.$$

Note that the relation (E2.5.1) is left–right symmetric and that all other five relations are clearly left–right symmetric. By symmetry let us only consider the first case. Let V_0 be the maximal subspace of V such that

$$V_0y_1 + V_0y_2 \subset WV = y_1V + y_2V + V^2.$$

By the assumption of the first case, $\dim V_0 \leqslant 1$. Write the six relations of B as

$$0 = f_i = f_{i1}x_1 + f_{i2}x_2 + f_{i3}y_1 + f_{i4}y_2$$

for $i = 1, \dots, 6$. Let $F = (f_{ij})_{6\times 4}$ be the matrix defined as in Lemma 2.1(d). The relation (E2.5.1) can be written as

$$0 = f_6 = f_{61}x_1 + f_{62}x_2 + f_{63}y_1 + f_{64}y_2$$

where $f_{63} = -y_2 + p_{11}y_1 + \tau_1$ and $f_{64} = p_{12}y_1 + \tau_2$. Since all other five relations are elements in $W \otimes V + V \otimes W$, we have $f_{i3}, f_{i4} \in V$ for all $i = 1, 2, 3, 4, 5$. Let $G = (g_{ij})_{4\times 6}$ be defined as in Lemma 2.1(e). Since $GF = 0$ [Lemma 2.1(g)], the third and fourth columns of this matrix equation imply that, for all $i = 1, 2, 3, 4$,

$$-\sum_{k=1}^{5} g_{ik}f_{k3} = g_{i6}(p_{12}y_1 + \tau_2) \in WV, \tag{E2.5.2}$$

$$-\sum_{k=1}^{5} g_{ik}f_{k4} = g_{i6}(-y_2 + p_{11}y_1 + \tau_1) \in WV. \tag{E2.5.3}$$

Since $p_{12} \neq 0$ and $\tau_2 \in V$, (E2.5.2) implies that $g_{i6}y_1 \in WV$. Since (E2.5.2) does not contain the term y_1y_2 and since (E2.5.1) has a non-zero term $p_{12}y_1y_2$, (E2.5.2) is a linear combination of relations other than (E2.5.1). So $g_{i6} \in V$. A linear combination of (E2.5.2) and (E2.5.3) shows that $g_{i6}y_2 \in WV$. By the definition of V_0, $g_{i6} \in V_0$ for all i, and hence the space spanned by the sixth column of G has dimension at most 1. This contradicts Lemma 2.2(c). Therefore part (a) follows.

(b) By part (a) and induction one sees that $V^n W = W V^n$ for all n. Hence $\sum_{n\geqslant 0} V^n W = \sum_{n\geqslant 0} W V^n$. Since $A = \sum_{n\geqslant 0} V^n$, we have $AW = WA$ or

$$Ay_1 + Ay_2 + A_{\geqslant 1} = y_1 A + y_2 A + A_{\geqslant 1}.$$

The assertion follows by adding k to the above equation.

(c) The subalgebra A has a quadratic relation and is a domain. By the proof of Lemma 2.4 for A', A is a regular algebra of dimension 2 with Hilbert series $H_A(t) = (1-t)^{-2}$.

By part (a), the relation (E2.5.1) can be simplified to the form of (R1), namely we may assume $\tau_1' = \tau_2' = 0$. We claim that

$$\textit{every element } f \textit{ in } B \textit{ can be written as an element in } \sum_{n_1, n_2 \geqslant 0} A y_1^{n_1} y_2^{n_2}.$$

Let $\deg_y$ be the degree of an element with respect to y_1 and y_2, namely, $\deg_y x_i = 0$ and $\deg_y y_i = 1$ for $i = 1, 2$. If $\deg_y f \leqslant 1$, the assertion follows from part (b). If $\deg_y f > 1$, the assertion follows from the induction on $\deg_y$, part (b) and the relation (R1) (which is equivalent to (E2.5.1) after we proved part (b)).

Since the Hilbert series of B is $(1-t)^{-4}$, an easy computation shows that $\sum_{n_1,n_2\geqslant 0} A y_1^{n_1} y_2^{n_2}$ is a free A-module with basis $\{y_1^{n_1} y_2^{n_2}\}_{\{n_1,n_2\geqslant 0\}}$. By Definition 1.3, B is a right double extension of A; so we can write $B = A_P[y_1, y_2; \sigma, \delta, \tau]$. By part (b) we have $Ay_1 \oplus Ay_2 \oplus A = y_1 A + y_2 A + A$. This implies that the Hilbert series of $y_1 A + y_2 A + A$ is equal to the Hilbert series of $Ay_1 \oplus Ay_2 \oplus A$, which is $(1+2t)(1-t)^{-2}$. By k-dimension counting, $y_1 A + y_2 A + A$ must be free over A with basis $\{1, y_1, y_2\}$. Hence

$$Ay_1 \oplus Ay_2 \oplus A = Ay_1 + Ay_2 + A = y_1 A + y_2 A + A = y_1 A \oplus y_2 A \oplus A.$$

By [28, Lemma 1.9], σ is invertible. Since $p_{12} \neq 0$, by [28, Proposition 1.13], B is a double extension. □

In Lemma 2.5 we assumed that $p_{12} \neq 0$. We show next that this condition is not too restrictive.

Lemma 2.6. *Let B be a regular domain of type* (14641) *generated by x_1, x_2, y_1, y_2. Suppose B has six quadratic relations satisfying the following conditions*:

(i) *The first relation is $f_1 = 0$ for some $f_1 \in V \otimes V$.*
(ii) *There are four relations in $V \otimes W + W \otimes V$ such that*
  (iia) *two of these are of the form $f_2' = 0$, $f_3' = 0$ where $f_2' = y_2 x_1 - h_2$ and $f_3' = y_2 x_2 - h_3$ for some $h_2, h_3 \in V \otimes W$; and*
  (iib) *the other two are $f_i' = 0$ for $i = 4, 5$ where $f_i \in V \otimes W + W \otimes V$.*
(iii) *The last relation f_6 is of the form*

$$-y_2 y_1 + p_{12} y_1 y_2 + p_{11} y_1^2 + h = 0$$

for some $h \in W \otimes V + V \otimes W$.

Then B cannot have three relations $f_i = 0$ for linearly independent elements $\{f_1, f_2, f_3\} \subset W \otimes V$.

Proof. If $p_{12} \neq 0$, by Lemma 2.5, B is a double extension and σ is invertible. In this case, it is easy to see that there is only one relation $f_1 \in W \otimes V$.

In the rest of the proof, we assume $p_{12}=0$. We continue to use the notations introduced in Lemma 2.1.

Assume on the contrary that three of the six relations of B are of the form $f_i=0$ where $f_i \in W \otimes V$ for $i=1,2,3$. This implies that the matrix F is of the form

$$\begin{pmatrix} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & f_{43} & * \\ \bullet & \bullet & f_{53} & * \\ \bullet & \bullet & -y_2+p_{11}y_1+q_3 & * \end{pmatrix}$$

where $\bullet$ denotes elements in W, and $*$ denotes elements in V, and $f_{43}, f_{53}, q_3 \in V$. The third column of the matrix equation $GF=0$ gives rise to the equations

$$g_{i4}f_{43}+g_{i5}f_{53}+g_{i6}(-y_2+p_{11}y_1+q_3)=0 \tag{E2.6.1}$$

for $i=1,2,3,4$. Since we assume $p_{12}=0$, any quadratic relation of B contains neither y_1y_2 nor y_2^2. This implies that $g_{i6} \in V$ for all i. By Lemma 2.2(c), the vector space V' spanned by g_{i6} for $i=1,\dots,4$ has dimension at least 2, whence $V'=V$. This means that there are at least two relations which can be derived from (E2.6.1) with $g_{i6} \neq 0$. Up to linear transformation we may assume that the relations derived from (E2.6.1) are of the form

$$h_4+x_1(-y_2+p_{11}y_1+q_3)=0 \quad \text{and} \quad h_5+x_2(-y_2+p_{11}y_1+q_3)=0$$

where $h_4, h_5 \in W \otimes V$. Denote these two relations as $f_4=0$ and $f_5=0$. Clearly, $\{f_4, f_5\}$ is linearly independent in the quotient space $(V \otimes W + W \otimes V)/W \otimes V$. Recall that, for the first three relations $f_i=0$, $i=1,2,3$, we have $f_i \in W \otimes V$ and, for the sixth relation, f_6 contains a non-zero monomial y_2y_1. Hence $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ are linearly independent, and we may use these relations as the defining relations of B. Hence the matrix F becomes

$$\begin{pmatrix} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & p_{11}x_1 & -x_1 \\ \bullet & \bullet & p_{11}x_2 & -x_2 \\ \bullet & \bullet & -y_2+p_{11}y_1+q_3 & f_{64} \end{pmatrix}$$

where $f_{64} \in V$. The third and fourth columns of the matrix equation $GF=0$ read as follows:

$$g_{i4}p_{11}x_1+g_{i5}p_{11}x_2+g_{i6}(-y_2+p_{11}y_1+q_3)=0$$

and

$$g_{i4}(-x_1)+g_{i5}(-x_2)+g_{i6}f_{64}=0.$$

Combining these two we obtain

$$g_{i6}(-y_2+p_{11}y_1+q_3+p_{11}f_{64})=0$$

which contradicts to the fact B is a domain. Therefore we have proved that if $p_{12}=0$ then B cannot have three relations $f_1=0$, $f_2=0$, $f_3=0$ with linearly independent elements $\{f_1, f_2, f_3\} \subset W \otimes V$. $\square$

Lemma 2.7. *Let B be as in Lemma 2.6. Then*

(a) $p_{12} \neq 0$.
(b) *B is a double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ where A is the subalgebra generated by x_1 and x_2.*

Proof. (a) Suppose on the contrary that $p_{12}=0$. We use the form of the six relations given in (i), (ii) and (iii) of Lemma 2.6.

Without loss of generality, we may assume that the terms y_2x_1, y_2x_2 do not appear in f_i for all $i \neq 2, 3$. Thus we can write F as follows:

$$F = \begin{pmatrix} * & * & 0 & 0 \\ y_2+q_1 & * & * & * \\ * & y_2+q_2 & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & -y_2+p_{11}y_1+q_3 & * \end{pmatrix}$$

where $q_i \in V$ and where all $*$ denotes elements in $V + ky_1$. For each i, we consider the equation

$$0 = \sum_{k=1}^{6} g_{ik} f_{k3} = \sum_{k=1}^{5} g_{ik} f_{k3} + g_{i6} f_{63}$$

which comes from the third column of the matrix equation $GF=0$. Since y_1y_2 and y_2^2 do not appear in any of the relations and since f_{k3} does not contain y_2 for all $k \neq 6$, g_{i6} does not contain either y_1 or y_2. So $g_{i6} \in V$ for all i. Similarly one can show that $g_{i2}, g_{i3} \in V$ for all i. The six relations of B can also be obtained by the multiplication $(x_1', x_2', x_3', x_4')G$. So the three relations corresponding to columns $2, 3, 6$ of $(x_1', x_2', x_3', x_4')G$ are in $W \otimes V$. But this contradicts Lemma 2.6. Therefore $p_{12} \neq 0$.

(b), (c) follows from part (a) and Lemma 2.5(c). $\square$

Now we can prove the main result in this section.

Theorem 2.8. *Let B be a regular domain of type* (14641)*. Suppose one of the following conditions holds*:

(i) *B is a right double extension.*
(ii) *There are $x_1, x_2 \in B_1$ such that*
  (iia) *B has a quadratic relation involving only x_1, x_2, and*
  (iib) *$B/(x_1, x_2)$ is regular of dimension* 2.

Then B is a double extension of a regular subalgebra of dimension 2.

Proof. (a) By Lemma 2.4 A is regular of dimension 2. Since B is a right double extension, hypotheses (i), (iia), (iib), (iii) of Lemma 2.6 hold. The assertion follows from Lemma 2.7.

(b) We would like to check (i), (ii), (iii) of Lemma 2.5. (i) is clear. (iii) follows from the fact that $B/(x_1, x_2)$ is regular of dimension 2. Since $B/(x_1, x_2)$ has only one relation, all other relations are $f_i = 0$ with $f_i \in V \otimes W + W \otimes V$. Thus we verified (ii) of Lemma 2.5. By Lemma 2.5, B is a double extension. $\square$

Proof of Proposition 0.3. If B is a double extension of A, by [28, Proposition 1.14], there is an algebra homomorphism $B \to B/(A_{\geqslant 1})$ and $B/(A_{\geqslant 1})$ is isomorphic to $k\langle y_1, y_2\rangle/(y_2y_1 - p_{12}y_1y_2 - p_{11}y_1^2)$. Since $p_{12} \neq 0$, $B/(A_{\geqslant 1})$ is regular of dimension 2. This is one implication. The other implication is Theorem 2.8. $\square$

To conclude this section we prove Theorem 0.2.

Theorem 2.9. *Suppose that B is a noetherian regular algebra of type* (14641) *and that B has a $\mathbb{Z}^2$-grading such that $B_1 = B_{01} \oplus B_{10}$ with both B_{01} and B_{10} non-zero. Then B is either a double extension or an Ore extension $A[x; \sigma]$ for some regular algebra A of dimension three.*

Proof. By [4, Theorem 3.9], B is a domain. Hence it is a quantum polynomial ring in the sense of [7, Definition 1.12].

By [7, Proposition 3.5], both subalgebras $B_{\mathbb{Z}\otimes 0}$ and $B_{0\times\mathbb{Z}}$ are Koszul noetherian regular domains of dimension strictly smaller than four. If $\dim B_{10} = 3$ and $\dim B_{01} = 1$, then $A := B_{\mathbb{Z}\otimes 0}$ is regular of dimension three and $C := B_{0\times\mathbb{Z}}$ is regular of dimension 1. It is straightforward to show that there is an exact sequence of algebras

$$0 \to A \to B \to C \to 0$$

satisfying Definition 1.1(a)–(d). Thus B is 1-extension of A. By Lemma 1.2(c), $B = A[x; \sigma, \delta]$. Since B is $\mathbb{Z}^2$-graded, $\delta = 0$.

If $\dim B_{10} = 1$ and $\dim B_{01} = 3$, a similar proof works.

It remains to consider $\dim B_{10} = \dim B_{01} = 2$. By [7, Proposition 3.5], both subalgebras $B_{\mathbb{Z}\otimes 0}$ and $B_{0\times\mathbb{Z}}$ are Koszul and regular of dimension 2 and $B_{\mathbb{Z}\otimes 0} \cong B/((B_{0\times\mathbb{Z}})_{\geqslant 1})$. So we are in the situation of Theorem 2.8(ii) and hence B is a double extension. $\square$

3. System C

The goal of Sections 3 and 4 is to classify regular domains of dimension four of the form $A_P[y_1, y_2; \sigma]$ up to isomorphism, or equivalently, to classify (P, σ) up to some equivalence relation. This is the first step toward a more complete (but not finished) classification of $A_P[y_1, y_2; \sigma, \delta, \tau]$. As explained in the introduction, we are interested in double extensions that are not iterated Ore extensions.

Since the base field k is algebraically closed, A is isomorphic to $k_Q[x_1, x_2]$ (see Lemma 2.4), which is either $k_q[x_1, x_2]$ with the relation $x_2x_1 = qx_1x_2$ (in this case $Q = (q, 0)$) or $k_J[x_1, x_2]$ with the relation $x_2x_1 = x_1x_2 + x_1^2$ (in this case $Q = J = (1, 1)$). To state some results uniformly we write A as $k_Q[x_1, x_2]$ where, by definition, $Q = (q_{12}, q_{11})$ and $k_Q[x_1, x_2] = k\langle x_1, x_2\rangle/(x_2x_1 = q_{11}x_1^2 + q_{12}x_1x_2)$. But in the computation we will set Q to be either $(1, 1)$ or $(q, 0)$.

Fix an A as in the last paragraph. Let $\sigma : A \to M_2(A)$ be a graded algebra homomorphism. Write

$$\sigma_{ij}(x_s) = \sum_{t=1}^{2} a_{ijst}x_t \tag{E3.0.1}$$

for all $i, j, s = 1, 2$ and where $a_{ijst} \in k$.

Using (E3.0.1) we can re-write the relation (R2) of the algebra $A_P[y_1, y_2; \sigma]$ as follows (note that in this case $\delta = 0$). Setting $r = x_1$ and x_2 in (R2), we have the following four relations:

$$\begin{aligned} y_1x_1 &= \sigma_{11}(x_1)y_1 + \sigma_{12}(x_1)y_2 \\ &= a_{1111}x_1y_1 + a_{1112}x_2y_1 + a_{1211}x_1y_2 + a_{1212}x_2y_2, \end{aligned} \tag{MR11}$$

$$\begin{aligned} y_1x_2 &= \sigma_{11}(x_2)y_1 + \sigma_{12}(x_2)y_2 \\ &= a_{1121}x_1y_1 + a_{1122}x_2y_1 + a_{1221}x_1y_2 + a_{1222}x_2y_2, \end{aligned} \tag{MR12}$$

$$\begin{aligned} y_2x_1 &= \sigma_{21}(x_1)y_1 + \sigma_{22}(x_1)y_2 \\ &= a_{2111}x_1y_1 + a_{2112}x_2y_1 + a_{2211}x_1y_2 + a_{2212}x_2y_2, \end{aligned} \tag{MR21}$$

$$\begin{aligned} y_2x_2 &= \sigma_{21}(x_2)y_1 + \sigma_{22}(x_2)y_2 \\ &= a_{2121}x_1y_1 + a_{2122}x_2y_1 + a_{2221}x_1y_2 + a_{2222}x_2y_2. \end{aligned} \tag{MR22}$$

We call the above 4 relations *mixing relations* between x_i and y_i. The double extension $A_P[y_1, y_2; \sigma]$ also has two non-mixing relations:

$$x_2x_1 = q_{12}x_1x_2 + q_{11}x_1^2, \tag{NRx}$$

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2. \tag{NRy}$$

Let Σ_{ij} be the matrix

$$\begin{pmatrix} a_{ij11} & a_{ij12} \\ a_{ij21} & a_{ij22} \end{pmatrix}$$

and let

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} a_{1111} & a_{1112} & a_{1211} & a_{1212} \\ a_{1121} & a_{1122} & a_{1221} & a_{1222} \\ a_{2111} & a_{2112} & a_{2211} & a_{2212} \\ a_{2121} & a_{2122} & a_{2221} & a_{2222} \end{pmatrix}. \tag{E3.0.2}$$

Since we assume that σ is a graded algebra homomorphism, σ is uniquely determined by Σ. Another matrix closely related to Σ is the following. Let M_{ij} be the matrix

$$\begin{pmatrix} a_{11ij} & a_{12ij} \\ a_{21ij} & a_{22ij} \end{pmatrix}$$

and let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

The matrix M is obtained by re-arranging the entries of Σ. Sometimes it is convenient to use M instead of Σ when we make a linear transformation of $\{y_1, y_2\}$. An easy linear algebra exercise shows that Σ is invertible if and only if M is invertible.

Since σ is an algebra homomorphism, we have, for i, j, f, g,

$$\begin{aligned} \sigma_{ij}(x_fx_g) &= \sum_{p=1}^{2} \sigma_{ip}(x_f)\sigma_{pj}(x_g) \\ &= \sum_{p,s,t=1}^{2} (a_{ipfs}x_s)(a_{pjgt}x_t) = \sum_{p,s,t=1}^{2} (a_{ipfs}a_{pjgt})x_sx_t \end{aligned}$$

$$= \left(\sum_{p=1}^{2} a_{ipf1}a_{pjg1}\right)x_1^2 + \left(\sum_{p=1}^{2} a_{ipf1}a_{pjg2}\right)x_1x_2 + \left(\sum_{p=1}^{2} a_{ipf2}a_{pjg1}\right)x_2x_1 + \left(\sum_{p=1}^{2} a_{ipf2}a_{pjg2}\right)x_2^2.$$

Using the relation $x_2x_1 = q_{11}x_1^2 + q_{12}x_1x_2$ in A, we obtain

$$\begin{aligned}\sigma_{ij}(x_f x_g) = {} & \big[(a_{i1f1}a_{1jg1} + a_{i2f1}a_{2jg1}) + q_{11}(a_{i1f2}a_{1jg1} + a_{i2f2}a_{2jg1})\big]x_1^2 \\ & + \big[(a_{i1f1}a_{1jg2} + a_{i2f1}a_{2jg2}) + q_{12}(a_{i1f2}a_{1jg1} + a_{i2f2}a_{2jg1})\big]x_1x_2 \\ & + (a_{i1f2}a_{1jg2} + a_{i2f2}a_{2jg2})x_2^2. \end{aligned} \tag{E3.0.3}$$

Since $x_2x_1 = q_{11}x_1^2 + q_{12}x_1x_2$ and since each σ_{ij} is a k-linear map,

$$\sigma_{ij}(x_2x_1) = q_{11}\sigma_{ij}(x_1x_1) + q_{12}\sigma_{ij}(x_1x_2) \tag{E3.0.4}$$

for all $i, j = 1, 2$. Now by (E3.0.3) we can express the left-hand and the right-hand sides of (E3.0.4) as polynomials of x_1 and x_2. By comparing coefficients of x_1^2, x_1x_2 and x_2^2 respectively, we obtain the following identities. The coefficients of x_1^2 of (E3.0.4) give rise to a constraint between coefficients

$$\begin{aligned} & (a_{i121}a_{1j11} + a_{i221}a_{2j11}) + q_{11}(a_{i122}a_{1j11} + a_{i222}a_{2j11}) \\ & = q_{11}\big[(a_{i111}a_{1j11} + a_{i211}a_{2j11}) + q_{11}(a_{i112}a_{1j11} + a_{i212}a_{2j11})\big] \\ & \quad + q_{12}\big[(a_{i111}a_{1j21} + a_{i211}a_{2j21}) + q_{11}(a_{i112}a_{1j21} + a_{i212}a_{2j21})\big]. \end{aligned} \tag{C1ij}$$

The letter C in (C1ij) stands for the Constraints on the Coefficients. The coefficients of x_1x_2 of (E3.0.4) give rise to

$$\begin{aligned} & (a_{i121}a_{1j12} + a_{i221}a_{2j12}) + q_{12}(a_{i122}a_{1j11} + a_{i222}a_{2j11}) \\ & = q_{11}\big[(a_{i111}a_{1j12} + a_{i211}a_{2j12}) + q_{12}(a_{i112}a_{1j11} + a_{i212}a_{2j11})\big] \\ & \quad + q_{12}\big[(a_{i111}a_{1j22} + a_{i211}a_{2j22}) + q_{12}(a_{i112}a_{1j21} + a_{i212}a_{2j21})\big]. \end{aligned} \tag{C2ij}$$

The coefficients of x_2^2 of (E3.0.4) give rise to

$$\begin{aligned} & (a_{i122}a_{1j12} + a_{i222}a_{2j12}) \\ & \quad = q_{11}(a_{i112}a_{1j12} + a_{i212}a_{2j12}) + q_{12}(a_{i112}a_{1j22} + a_{i212}a_{2j22}). \end{aligned} \tag{C3ij}$$

Next we apply (R3.1), (R3.2) and (R3.3) to the elements $r = x_1$ and x_2, and obtain more relations between a_{ijkl}. For $i, f, g, s, t = 1, 2$,

$$\begin{aligned}\sigma_{fg}\big(\sigma_{st}(x_i)\big) &= \sigma_{fg}\left(\sum_{w=1}^{2} a_{stiw}x_w\right) = \sum_{w=1}^{2} a_{stiw}\sigma_{fg}(x_w)\\ &= \sum_{w=1}^{2} a_{stiw}\sum_{j=1}^{2} a_{fgwj}x_j = \sum_{j=1}^{2}(a_{sti1}a_{fg1j}+a_{sti2}a_{fg2j})x_j\\ &= (a_{sti1}a_{fg11}+a_{sti2}a_{fg21})x_1+(a_{sti1}a_{fg12}+a_{sti2}a_{fg22})x_2. \qquad \text{(E3.0.5)}\end{aligned}$$

Recall that $P=(p_{12},p_{11})$; and we will set $P=(1,1)$ or $(p,0)$ when we do the computation later. Using (E3.0.5) relations (R3.1)–(R3.3) (when applied to x_i) are equivalent to the following constraints on coefficients:

$$\begin{aligned}&(a_{11i1}a_{211j}+a_{11i2}a_{212j})+p_{11}(a_{11i1}a_{221j}+a_{11i2}a_{222j})\\ &\quad= p_{11}(a_{11i1}a_{111j}+a_{11i2}a_{112j})+p_{11}^2(a_{11i1}a_{121j}+a_{11i2}a_{122j})\\ &\qquad+p_{12}(a_{21i1}a_{111j}+a_{21i2}a_{112j})+p_{11}p_{12}(a_{21i1}a_{121j}+a_{21i2}a_{122j}), \qquad \text{(C4ij)}\\ &(a_{12i1}a_{211j}+a_{12i2}a_{212j})+p_{12}(a_{11i1}a_{221j}+a_{11i2}a_{222j})\\ &\quad= p_{11}(a_{12i1}a_{111j}+a_{12i2}a_{112j})+p_{11}p_{12}(a_{11i1}a_{121j}+a_{11i2}a_{122j})\\ &\qquad+p_{12}(a_{22i1}a_{111j}+a_{22i2}a_{112j})+p_{12}^2(a_{21i1}a_{121j}+a_{21i2}a_{122j}), \qquad \text{(C5ij)}\\ &(a_{12i1}a_{221j}+a_{12i2}a_{222j})\\ &\quad= p_{11}(a_{12i1}a_{121j}+a_{12i2}a_{122j})+p_{12}(a_{22i1}a_{121j}+a_{22i2}a_{122j}). \qquad \text{(C6ij)}\end{aligned}$$

Note that there is a symmetry between the first three C-constraints ((C1ij), (C2ij), (C3ij)) and the last three C-constraints ((C4ij), (C5ij), (C6ij)). The following proposition summarizes the above analysis.

Proposition 3.1.

(a) *Let $A_P[y_1,y_2;\sigma]$ be a right double extension and suppose the data $\{\Sigma,P,Q\}$ are defined as above corresponding to the k-linear base $\{x_1,x_2,y_1,y_2\}$. Then the six equations* (C1ij)–(C6ij) *are satisfied. Further,* $\det\Sigma\neq 0$ *if and only if $A_P[y_1,y_2;\sigma]$ is a double extension.*

(b) *Suppose a matrix Σ as in* (E3.0.2), *and parameter sets $P=(p_{12},p_{11})$ and $Q=(q_{12},q_{11})$ with $p_{12}q_{12}\neq 0$ are given. If the six equations* (C1ij)–(C6ij) *hold and* $\det\Sigma\neq 0$, *then the six relations* (MR11), (MR12), (MR21), (MR22), (NRx), *and* (NRy) *define a double extension $A_P[y_1,y_2;\sigma]$.*

Proof. (a) First of all, if $A_P[y_1,y_2;\sigma]$ is a right double extension, the matrix Σ can be defined in the same way. By the analysis given above, the six equations (C1ij)–(C6ij) are satisfied.

One easily sees that the matrix Σ is invertible if and only if $y_1V+y_2V=Vy_1+Vy_2$ where $V=kx_1+kx_2$. By induction on the degree of elements in A, the latter is equivalent to $y_1A+y_2A=Ay_1+Ay_2$, and hence equivalent to $y_1A+y_2A+A=Ay_1+Ay_2+A$. Therefore Σ is invertible if and only if $y_1A+y_2A+A=Ay_1+Ay_2+A$. By [28, Lemma 1.9], the equation $y_1A+y_2A+A=Ay_1+Ay_2+A$ holds if and only if σ is invertible. Combining [28, Lemma 1.9] with [28, Proposition 1.13], σ is invertible if and only if $A_P[y_1,y_2;\sigma]$ is a double extension. Therefore the last assertion follows.

(b) By [28, Proposition 1.11], if the coefficients $\{a_{ijfg}\}$ (the entries of Σ) satisfy the six C-constraints, then the six quadratic relations (MR11)–(MR22), (NRx) and (NRy) define a right double extension $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$. By part (a), $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$ is a double extension if and only if the matrix Σ is invertible. $\square$

By **System C** we mean the system of the six equations (C1ij)–(C6ij) together with $\det \Sigma \neq 0$. In the next section we will first fix $P = (p_{12}, p_{11})$ and $Q = (q_{12}, q_{11})$. A **solution to System C** or a **C-solution** is a matrix Σ with entries a_{ijst} satisfying System C.

Next we introduce some equivalence relations between C-solutions. If we change the basis $\{x_1, x_2\}$ to $x_1' = b_{11}x_1 + b_{12}x_1$, $x_2' = b_{21}x_1 + b_{22}x_2$, then the matrix Σ is changed to a new Σ. The following lemma is clear.

Lemma 3.2. *Let* $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *and* $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

(a) *The* 4 *mixing relations* (MR11)–(MR22) *can be written as*

$$y_i X = y_i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \Sigma_{i1} \begin{pmatrix} x_1 y_1 \\ x_2 y_1 \end{pmatrix} + \Sigma_{i2} \begin{pmatrix} x_1 y_2 \\ x_2 y_2 \end{pmatrix} = \Sigma_{i1} X y_1 + \Sigma_{i2} X y_2$$

for $i = 1, 2$.

(b) *If* X *is changed to* $X' = BX$ *where* $B = (b_{ij})_{2\times 2}$ *is an invertible matrix, then* Σ' *is equal to* $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \Sigma \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.

(c) *Suppose* $Q = (1, 0)$ (*in this case the algebra* A *is the commutative ring* $k[x_1, x_2]$). *After a linear transformation of* X, *we may assume either that* $a_{1212} = a_{1221} = 0$ *or that* $a_{1212} = 0$, $a_{1221} = 1$.

(d) *The* 4 *mixing relations can also be written as*

$$Y x_i = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} x_i = M_{i1} \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \end{pmatrix} + M_{i2} \begin{pmatrix} x_2 y_1 \\ x_2 y_2 \end{pmatrix} = M_{i1} x_1 Y + M_{i2} x_2 Y$$

for $i = 1, 2$.

(e) *If* Y *is changed to* $Y' = BY$ *where* $B = (b_{ij})_{2\times 2}$ *is an invertible matrix, then* M' *is equal to* $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} M \begin{pmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}$.

Lemma 3.3. *Let* Σ *be a C-solution and let* $B = (k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ *where* σ *is determined by* Σ. *Let* $0 \neq h \in k$.

(a) *B is a $\mathbb{Z}^2$-graded algebra. Let $\gamma : x_i \to x_i$, $y_i \to h y_i$. Then γ extends to a graded automorphism of B.*

(b) *$h\Sigma$ is a C-solution. Let σ' be the algebra automorphism determined by $h\Sigma$. Then $B' := (k_Q[x_1, x_2])_P[y_1, y_2; \sigma']$ is a graded twist of B by γ in the sense of* [26].

Proof. (a) The assertion is clear.

(b) Since the relations (Csij) are homogeneous and since $\det(h\Sigma) = h^4 \det \Sigma$, Σ is a C-solution if and only if $h\Sigma$ is. The second assertion can be verified by working on the relations of the twist B^γ. □

In general the algebra B and its twist B^γ are not isomorphic as algebras. However these algebras have many common properties since the category of graded B-modules is equivalent to the category of graded B^γ-modules (see [26]).

Definition 3.4.

(a) We say Σ and Σ' are *twist equivalent* if $\Sigma' = h\Sigma$ for some $0 \neq h \in k$. In this case we say Σ' is a *twist* of Σ. By Lemma 3.3(b), we can replace Σ by its twists (without changing Q and P) to obtain another double extension.

(b) We say (Σ, Q, P) and (Σ', Q', P') are *linearly equivalent* if there is a graded algebra isomorphism from $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ to $(k_{Q'}[x'_1, x'_2])_{P'}[y'_1, y'_2; \sigma']$ mapping $kx_1 + kx_2 \to kx'_1 + kx'_2$ and $ky_1 + ky_2 \to ky'_1 + ky'_2$. Using this isomorphism we can pull back x'_i and y'_i to the algebra $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$; then we can assume that $\{x'_1, x'_2\}$ (respectively, $\{y'_1, y'_2\}$) is another basis of $\{x_1, x_2\}$ (respectively, $\{y_1, y_2\}$). In case $Q = Q'$ and $P = P'$, then we just say that Σ and Σ' are *linearly equivalent*.

(c) We say (Σ, Q, P) and (Σ', Q', P') are *equivalent* if (Σ, Q, P) and $(h\Sigma', Q', P')$ are linearly equivalent for some $0 \neq h \in k$.

The following lemma is clear.

Lemma 3.5.

(a) *Twist equivalence is an equivalence relation.*
(b) *Linear equivalence is an equivalence relation.*
(c) *Equivalence between (Σ, Q, P) and (Σ', Q', P') defined above is an equivalence relation.*

Since our goal is to classify $A_P[y_1, y_2; \sigma]$ up to isomorphism (or even up to twist), we will classify Σ up to (linear) equivalence. Here are the strategies before we move into complicated computations in the next section:

Strategies

Strategy 1. We will use the mathematical software Maple as much as possible to reduce the length of the computations. The process of solving System C by Maple and the corresponding code will be omitted since the code is very simple. Not all solutions will be listed here since the list of the solutions to System C is still large. We need to do more reductions in the next two steps to achieve our final solution. Even then we will see a large number of solutions.

Strategy 2. We will try not to analyze iterated Ore extensions. In many cases, when a C-solution gives rise to an iterated Ore extension, we will stop. To test when a Σ gives rise an iterated Ore extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ we will mainly use Proposition 3.6 below. Most of such solutions will not be listed; but a few examples will be given in Section 4.1.

Strategy 3. Further reductions will be done by using equivalence relations between (Σ, Q, P) and (Σ', Q', P'). There is no unique way of doing this since linear equivalence is dependent on particular choices of Q and P. This is one of the reasons we break the computation into four cases in the next four subsections according to the form of P.

Proposition 3.6. *Let B be a double extension $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$.*

(a) *If $\Sigma_{12} = 0$, then B is an iterated Ore extension.*
(b) *If $\Sigma_{21} = 0$ and $p_{11} = 0$, then B is an iterated Ore extension.*

Proof. (a) If $\Sigma_{12} = 0$, then $\sigma_{12} = 0$. Hence the first half of the relation (R2) becomes

$$y_1 r = \sigma_{11}(r) y_1 + \delta_1(r)$$

for all $r \in A := k_Q[x_1, x_2]$. It is easy to check that σ_{11} is an automorphism of A and δ_1 is a σ-derivation of A. Therefore the subalgebra generated by x_1, x_2, y_1 is an Ore extension of A. The second half of (R2) together with (R1) shows that B is an Ore extension of $A[y_1; \sigma_{11}, \delta_1]$. The assertion follows.

(b) Since $p_{11}=0$, we switch y_1 and y_2 without changing the form of (R1), and then $\Sigma_{21}=0$ becomes $\Sigma_{12}=0$. The assertion follows from (a). Note that if $p_{11}\neq 0$, then we cannot switch y_1 and y_2 to keep the form of (R1). □

The following proposition is a consequence of the above.

Proposition 3.7. *Let B be a trimmed double extension $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$ where σ is determined by the matrix Σ.*

(a) *Considering $A' := k_P[y_1,y_2]$ as the subring and $\{x_1,x_2\}$ as the set of generators over A', B is a double extension $(k_P[y_1,y_2])_Q[x_1,x_2;\alpha]$ where α is determined by the matrix M^{-1}.*
(b) *If $M_{12}=0$, then B is an iterated Ore extension of $k_P[y_1,y_2]$.*
(c) *If $M_{21}=0$ and $q_{11}=0$, then B is an iterated Ore extension of $k_P[y_1,y_2]$.*

Proof. (a) We need to switch the roles played by x_i and y_i. The four mixing relations can be written as

$$\begin{pmatrix} y_1x_1\\ y_2x_1\\ y_1x_2\\ y_2x_2\end{pmatrix} = M\begin{pmatrix} x_1y_1\\ x_1y_2\\ x_2y_1\\ x_2y_2\end{pmatrix}. \tag{E3.7.1}$$

This implies that

$$\begin{pmatrix} x_1y_1\\ x_1y_2\\ x_2y_1\\ x_2y_2\end{pmatrix} = M^{-1}\begin{pmatrix} y_1x_1\\ y_2x_1\\ y_1x_2\\ y_2x_2\end{pmatrix}.$$

Therefore B is the double extension $(k_P[y_1,y_2])_Q[x_1,x_2;\alpha]$ where α is determined by the matrix M^{-1}.

(b), (c) By part (a), the matrix M^{-1} plays the role of Σ-matrix (if we switch x_i with y_i). Since $M_{12}=0$ (respectively, $M_{21}=0$) if and only if $(M^{-1})_{12}=0$ (respectively, $(M^{-1})_{21}=0$), the assertions follow from Proposition 3.6(a), (b). □

The matrix M also appears in a slightly different setting, see the next proposition. For any $P=(p_{12},p_{11})$, let P° denote the set $(p_{12}^{-1}, -p_{12}^{-1}p_{11})$.

Proposition 3.8. *The opposite ring of $B := (k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$ is a double extension $(k_{P^\circ}[y_1,y_2])_{Q^\circ}[x_1,x_2;\xi]$ where ξ is determined by the matrix M.*

Proof. Let $\star$ be the multiplication of the opposite ring B^{op}. The relation

$$y_2y_1 = p_{12}y_1y_2 + p_{11}y_1^2$$

in B implies the relation

$$y_2\star y_1 = p_{12}^{-1}y_1\star y_2 + \left(-p_{12}^{-1}p_{11}\right)y_1^2$$

in B^{op}. The same is true for the relation between x_1 and x_2. The relations (E3.7.1) in B imply the following relations in B^{op}:

$$\begin{pmatrix} x_1 \star y_1 \\ x_1 \star y_2 \\ x_2 \star y_1 \\ x_2 \star y_2 \end{pmatrix} = M \begin{pmatrix} y_1 \star x_1 \\ y_2 \star x_1 \\ y_1 \star x_2 \\ y_2 \star x_2 \end{pmatrix}.$$

Recall that x_i and y_i are switched in the double extension $(k_{P^\circ}[y_1, y_2])_{Q^\circ}[x_1, x_2; \xi]$. So the matrix M plays the role of the Σ-matrix for the homomorphism ξ. $\square$

4. A classification of $\{\Sigma, P, Q\}$

Now we start our classification.

4.1. *Case one:* $P = (1, 1)$

In this subsection we classify Σ which gives a solution to System C when $P = (1, 1)$. We consider the following subcases.

Subcase 4.1.1. $Q = (1, 1)$. System C is solved by Maple to give two solutions in this case.

Solution one: $\Sigma = \begin{pmatrix} f & 0 & 0 & 0 \\ g & f & 0 & 0 \\ h & 0 & f & 0 \\ m & h & g & f \end{pmatrix}$ where $f \neq 0$.

Solution two: $\Sigma = \begin{pmatrix} f & 0 & 0 & 0 \\ g & f & 0 & 0 \\ -f^2/g & 0 & f & 0 \\ h & -f(f-m+g)/g & m & f \end{pmatrix}$ where $fg \neq 0$.

In both solutions we have $\Sigma_{12} = 0$. By Proposition 3.6(a), these Σ will produce iterated Ore extensions. By Strategy 2, we will not study these algebras further in this paper.

Subcase 4.1.2. $Q = (q, 0)$ where $q \neq 0, \pm 1$. System C is solved by Maple to give one solution, which is $\Sigma = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ h & 0 & f & 0 \\ 0 & m & 0 & g \end{pmatrix}$ where $fg \neq 0$. Again since $\Sigma_{12} = 0$, by Proposition 3.6, we will only obtain an iterated Ore extension.

Subcase 4.1.3. $Q = (-1, 0)$. There are two C-solutions. One is the same as the solution in Subcase 4.1.2; so it gives rise to an iterated Ore extension. The other is $\Sigma = \begin{pmatrix} 0 & f & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & fh/g & 0 & f \\ h & 0 & g & 0 \end{pmatrix}$ where $fg \neq 0$. Again in this case we only obtain an iterated Ore extension. Up to this point we only used Strategies 1 and 2.

Subcase 4.1.4. $Q = (1, 0)$. System C is solved by Maple to give 15 solutions, 13 of which have the property $\Sigma_{12} = 0$. To save space we will not list these solutions. Next we use Strategy 3.

Since $Q = (1, 0)$, we can make a linear transformation of $\{x_1, x_2\}$. By Lemma 3.2(c) we may further assume that either $a_{1212} = a_{1221} = 0$ or $a_{1212} = 0$, $a_{1221} = 1$.

If $a_{1212} = 0$ and $a_{1221} = 0$, System C is solved by Maple to give 15 solutions, all of which have the property that $\Sigma_{12} = 0$. So we only consider the case when $a_{1212} = 0$ and $a_{1221} = 1$. System C is solved to give a single solution: $\Sigma = \begin{pmatrix} f & 0 & 0 & 0 \\ g & f & 1 & 0 \\ 0 & 0 & f & 0 \\ m & -2f & -g-1 & f \end{pmatrix}$ or equivalently $\Sigma = \begin{pmatrix} f & 0 & 0 & 0 \\ gf & f & f & 0 \\ 0 & 0 & f & 0 \\ mf & -2f & -gf-f & f \end{pmatrix}$ after using a linear transformation $X' = \begin{pmatrix} 1 & 0 \\ 0 & f^{-1} \end{pmatrix} X$. Up to a twist equivalence, we may assume that $\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ g & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ m & -2 & -g-1 & 1 \end{pmatrix}$. Now we will make a linear transformation of $Y = (y_1, y_2)^T$. It is a bit easier to see this by using the matrix M. The last Σ is equivalent to $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ g & 1 & 1 & 0 \\ m & -g-1 & -2 & 1 \end{pmatrix}$. Since $P = (1,1)$, we can change Y to $Y' = BY$ where $B = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$. By doing so the structure of the relations will not change, but the matrix M becomes $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ m' & -1 & -2 & 1 \end{pmatrix}$ where $m' = m + g + g^2$. Let g denote the new m'. We have $\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ g & -2 & -1 & 1 \end{pmatrix}$. Now we make another linear transformation $X' = BX$ with $B = \begin{pmatrix} 1 & 0 \\ g/2 & 1 \end{pmatrix}$, then we have $\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix}$. This is the only possible Σ up to (linear and twist) equivalence. Therefore up to linear equivalence, we have the first interesting case:

Algebra $\mathbb{A}$: $\Sigma = h \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix}$ and $M = h \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix}$ where $h \neq 0$; and $P = (1,1)$, $Q = (1,0)$. In the rest of the section let h be a non-zero scalar in k. We can easily write down the relations of the algebra $\mathbb{A}$ from $\{\Sigma, P, Q\}$. The matrix Σ gives us the four mixing relations between x_i and y_i. The Q tells us the relation between x_1 and x_2 and the P tells us the relation between y_1 and y_2. Here are the six quadratic relations of the algebra $\mathbb{A}$:

$$x_2x_1 = x_1x_2,$$
$$y_2y_1 = y_1y_2 + y_1^2,$$
$$y_1x_1 = x_1y_1,$$
$$y_1x_2 = x_2y_1 + x_1y_2,$$
$$y_2x_1 = x_1y_2,$$
$$y_2x_2 = -2x_2y_1 - x_1y_2 + x_2y_2.$$

All algebras in this section are generated by x_1, x_2, y_1 and y_2. To save space, we will not write down explicitly the relations of other algebras except for the algebra $\mathbb{Z}$ at the end of this section.

By Proposition 3.7(a) any double extension $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$ is isomorphic to $(k_P[y_1,y_2])_Q[x_1,x_2;\alpha]$ where α is determined by the matrix M^{-1}. In the case of the algebra $\mathbb{A}$, we have $M_{12} = 0$. By Proposition 3.7(b) $\mathbb{A}$ is an iterated Ore extension of $k_P[y_1,y_2]$. However, there are possible δ, τ such that $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma,\delta,\tau]$ is not an iterated Ore extension of any regular algebra of dimension 2. This is the reason the algebra $\mathbb{A}$ is not deleted from our 26 families.

The point-scheme of the algebra $\mathbb{A}$ can be computed. We see that the dimension of the point-scheme is 1 in this case and the details are omitted. Recall from [28, Section 2] that the determinant of σ is defined to be

$$\det\sigma = -p_{11}\sigma_{12}\sigma_{11} + \sigma_{22}\sigma_{11} - p_{12}\sigma_{12}\sigma_{21}$$

which is an algebra automorphism of $k_Q[x_1, x_2]$. For the algebra $\mathbb{A}$ we have

$$\det\sigma \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This is the end of classification of $\{\Sigma, Q\}$ when $P = (1, 1)$.

4.2. Case two: $P = (p, 0)$, $p \neq \pm 1$

We consider four subcases as in case one. Some arguments are similar to the ones given in case one above (Section 4.1), so most of the details will be omitted.

Subcase 4.2.1. $Q = (1, 1)$. There is only one C-solution in which $\Sigma_{12} = 0$. By Proposition 3.6(a) we only get an iterated Ore extension.

Subcase 4.2.2. $Q = (q, 0)$ where $q \neq \pm 1$.

The following lemma provides some information about possible parameter sets P and Q when the double extension associated to $\{\Sigma, P, Q\}$ is not an iterated Ore extension. The proof of the following lemma is based on tedious computation and can be verified by Maple very quickly. Since the lemma will not be used later, we decide to omit its proof.

Lemma 4.1. *Suppose $P = (p, 0)$ and $Q = (q, 0)$ where $p \neq \pm 1$ and $q \neq \pm 1$. Suppose Σ is a C-solution (in particular* $\det \Sigma \neq 0$*).*

(a) *If $p \neq \pm i, \xi_3, \xi_3^2$ where i is a primitive 4th root of 1 and ξ_3 is the primitive 3rd root of 1, then either $\Sigma_{12} = 0$ or $\Sigma_{21} = 0$.*
(b) *Suppose $\Sigma_{12} \neq 0$ and $\Sigma_{21} \neq 0$.*
   (i) *If $p^2 = -1$ (or $p = \pm i$), then either $q = p$ or $q = p^{-1}$.*
   (ii) *If $p = \xi_3$ or ξ_3^2, then either $q = p$ or $q = p^{-1}$.*
   (iii) *After exchanging x_1 and x_2, we may assume that $q = p$.*

The next lemma follows from Lemma 3.2.

Lemma 4.2. *We fix $P = (p, 0)$ and $Q = (q, 0)$. Let Σ be a C-solution with $\Sigma_{ij} = (a_{ijst})_{2\times 2}$.*

(a) *If the basis $\{x_1, x_2\}$ is changed to $\{x_1, ax_2\}$, then the entry a_{ijst} of Σ is changed to $a^{(s-t)} a_{ijst}$.*
(b) *If the basis $\{y_1, y_2\}$ is changed to $\{y_1, by_2\}$, then the entry a_{ijst} of Σ is changed to $b^{(i-j)} a_{ijst}$.*
(c) *If $a_{1211} \neq 0$, after a linear transformation of $\{y_1, y_2\}$, we may assume $a_{1211} = 1$.*
(d) *If $a_{1221} \neq 0$, after a linear transformation of $\{x_1, x_2\}$ (or $\{y_1, y_2\}$), we may assume $a_{1221} = 1$.*

According to Lemma 4.2 we may assume that $a_{1211} = 0$ or 1 and $a_{1221} = 0$ or 1. From now on we will only consider those solutions with $\Sigma_{12} \neq 0$. (If $\Sigma_{12} = 0$, then use Proposition 3.6(a).) We may further assume that the first column of Σ_{12} is non-zero, since if the second column is non-zero, then by switching x_1 with x_2 we obtain that the first column of Σ_{12} is non-zero. Hence there are three cases to consider (up to a linear equivalence):

Case (i): $a_{1211} = 1$, $a_{1221} = 0$. Maple gives no C-solution.

Case (ii): $a_{1211} = 0$, $a_{1221} = 1$. Maple gives two C-solutions. One of these has $\Sigma_{21} = 0$, so we omit this one by Proposition 3.6(b). The other is $\Sigma = \begin{pmatrix} 0 & 0 & 0 & f \\ 0 & 0 & 1 & 0 \\ 0 & -fg & 0 & 0 \\ g & 0 & 0 & 0 \end{pmatrix}$ with $fg \neq 0$, and $p = q$ and $p^2 = -1$. Using Lemma 4.2 above Σ is linearly equivalent to

Algebra $\mathbb{B}$: $\Sigma = h\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$; and $P = (p, 0) = Q$ and $p^2 = -1$.

Recall that P° denotes the set $(p_{12}^{-1}, -p_{12}^{-1}p_{11})$ where $P = (p_{12}, p_{11})$.

Definition 4.3. Let $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ and $(k_{Q'}[x_1, x_2])_{P'}[y_1, y_2; \sigma']$ be two double extensions, with corresponding matrices Σ and Σ' respectively.

(a) Two double extensions are called *Σ-M-dual* if the data $\{M, Q^\circ, P^\circ\}$ is equivalent to $\{\Sigma', P', Q'\}$ in the sense of Definition 3.4.
(b) A double extension is called *Σ-M-selfdual* if $\{M, Q^\circ, P^\circ\}$ is equivalent to $\{\Sigma, P, Q\}$ in the sense of Definition 3.4.

By Proposition 3.8, if $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ and $(k_{Q'}[x_1, x_2])_{P'}[y_1, y_2; \sigma']$ are Σ-M-dual, then

$$\big(k_Q[x_1, x_2]\big)_P[y_1, y_2; \sigma]^\gamma \cong \big(\big(k_{Q'}[x_1, x_2]\big)_{P'}[y_1, y_2; \sigma']\big)^{op}$$

for some automorphism γ. In particular, if $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ is Σ-M-selfdual then

$$\big(k_Q[x_1, x_2]\big)_P[y_1, y_2; \sigma]^\gamma \cong \big(k_Q[x_1, x_2]\big)_P[y_1, y_2; \sigma]^{op}$$

for some γ.

It is easy to verify that the algebra $\mathbb{B}$ is Σ-M-selfdual. Also the algebra $\mathbb{B}$ contains two cases with the same matrix Σ, namely, $p = i$ and $p = -i$.

Case (iii): $a_{1211} = 1$, $a_{1221} = 1$. There are four C-solutions such that $\Sigma_{21} \neq 0$. All four solutions are linearly equivalent in the sense of Definition 3.4. Here we use the linear equivalences of the form $(\Sigma, (p, 0), (q, 0)) \sim (\Sigma', (p', 0), (q', 0))$ where p' is p or p^{-1} and q' is q or q^{-1}. So up to linear equivalence, we only have one C-solution:

Algebra $\mathbb{C}$: $\Sigma = h\begin{pmatrix} -1 & p^2 & 1 & -p \\ -p & 1 & 1 & -p \\ -p & -2p^2 & p & -p \\ -p & p^2 & 1 & -1 \end{pmatrix}$ and $M = h\begin{pmatrix} -1 & 1 & p^2 & -p \\ -p & p & -2p^2 & -p \\ -p & 1 & 1 & -p \\ -p & 1 & p^2 & -1 \end{pmatrix}$; and $P = (p, 0) = Q$ and $p^2 + p + 1 = 0$.

It is easy to check that the algebra $\mathbb{C}$ is Σ-M-selfdual. Of course the equation $p^2 + p + 1 = 0$ has two solutions, and we may think of the algebra $\mathbb{C}$ as containing two different cases. This is the end of Subcase 4.2.2.

Subcase 4.2.3. $Q = (-1, 0)$. Similar to the argument given in Subcase 4.2.2 we need to consider the following three cases: $(a_{1211}, a_{1221}) = (1, 0)$ or $(a_{1211}, a_{1221}) = (0, 1)$ or $(a_{1211}, a_{1221}) = (1, 1)$.

Case (i): $a_{1211} = 1$, $a_{1221} = 0$. C-solutions have $\Sigma_{21} = 0$. So we stop here.

Case (ii): $a_{1211} = 0$, $a_{1221} = 1$. Only one C-solution has the property $\Sigma_{21} \neq 0$.

Algebra $\mathbb{D}$: $\Sigma = h\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p^2 & 1 & 0 \\ 0 & 0 & p & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 1 & -p^2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$; $P = (p, 0)$ where p is a general parameter which could be ± 1 and $Q = (-1, 0)$.

Case (iii): $a_{1211} = 1$, $a_{1221} = 1$. There are two C-solutions such that $\Sigma_{21} \neq 0$.

Algebra $\mathbb{E}$: $\Sigma = h\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$; $P = (p, 0)$ where $p^2 = -1$ and $Q = (-1, 0)$.

Algebra $\mathbb{F}$: $\Sigma = h\begin{pmatrix} -1 & -p & 1 & -1 \\ -p & 1 & 1 & 1 \\ -p & p & p & 1 \\ -p & -p & 1 & -p \end{pmatrix}$ and $M = h\begin{pmatrix} -1 & 1 & -p & -1 \\ -p & p & p & 1 \\ -p & 1 & 1 & 1 \\ -p & 1 & -p & -p \end{pmatrix}$; $P = (p, 0)$ where $p^2 = -1$ and $Q = (-1, 0)$.

Subcase 4.2.4. $Q = (1,0)$. We can make linear transformations of X such that Σ_{12} is one of the standard forms: $\Sigma_{12} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ or $\Sigma_{12} = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$. In particular we may assume either $(a_{1212}, a_{1221}) = (0,0)$ or $(a_{1212}, a_{1221}) = (0,1)$.

Case (i): $(a_{1212}, a_{1221}) = (0,0)$. There is no C-solution with $\Sigma_{21} \neq 0$.

Case (ii): $(a_{1212}, a_{1221}) = (0,1)$. There is only one C-solution such that $\Sigma_{21} \neq 0$.

Algebra $\mathbb{G}$: $\Sigma = h \begin{pmatrix} p & 0 & 0 & 0 \\ p & p^2 & 1 & 0 \\ 0 & 0 & p & 0 \\ f & 0 & -1 & 1 \end{pmatrix}$ and $M = h \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ p & 1 & p^2 & 0 \\ f & -1 & 0 & 1 \end{pmatrix}$ with $f \neq 0$; $P = (p,0)$ where p is general and $Q = (1,0)$.

This is the end of Section 4.2 where $P = (p,0)$ and $p \neq 0, \pm 1$.

4.3. *Case three:* $P = (-1,0)$.

Following the steps as before we consider four subcases.

Subcase 4.3.1. $Q = (1,1)$. Up to linear equivalence, System C has one solution with $\Sigma_{12} \neq 0$:

Algebra $\mathbb{H}$: $\Sigma = h \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & f & 1 \\ 1 & 0 & 0 & 0 \\ f & 1 & 0 & 0 \end{pmatrix}$ and $M = h \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f & 0 & 1 \\ f & 0 & 1 & 0 \end{pmatrix}$; $P = (-1,0)$ and $Q = (1,1)$.

Subcase 4.3.2. $Q = (q,0)$ where $q \neq \pm 1$. The system C has four solutions up to linear equivalence with $\Sigma_{12} \neq 0$ and $\Sigma_{21} \neq 0$:

Algebra $\mathbb{I}$: $\Sigma = h \begin{pmatrix} -q & -q & 1 & -q \\ 1 & 1 & 1 & -q \\ 1 & q & q & -q \\ -1 & -q & 1 & -1 \end{pmatrix}$ and $M = h \begin{pmatrix} -q & 1 & -q & -q \\ 1 & q & q & -q \\ 1 & 1 & 1 & -q \\ -1 & 1 & -q & -1 \end{pmatrix}$; $P = (-1,0)$ and $Q = (q,0)$ where $q^2 = -1$. Algebra $\mathbb{I}$ is Σ-M-dual to algebra $\mathbb{F}$.

Algebra $\mathbb{J}$: $\Sigma = h \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$; $P = (-1,0)$ and $Q = (q,0)$ where $q^2 = -1$. Algebra $\mathbb{J}$ is Σ-M-dual to algebra $\mathbb{E}$.

Algebra $\mathbb{K}$: $\Sigma = h \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & f & 0 & 0 \end{pmatrix}$ and $M = h \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & f & 0 \end{pmatrix}$ where $f \neq 0$; $P = (-1,0)$ and $Q = (q,0)$ where q is a general parameter which could be ± 1.

Algebra $\mathbb{L}$: $\Sigma = h \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \\ f & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $M = h \begin{pmatrix} 0 & f & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ where $f \neq 0$; $P = (-1,0)$ and $Q = (q,0)$ where q is a general parameter which could be ± 1.

Subcase 4.3.3. $Q = (-1,0)$. There are nine C-solutions up to linear equivalence that have $\Sigma_{12} \neq 0$ and $\Sigma_{21} \neq 0$.

Algebra $\mathbb{M}$: $\Sigma = h \begin{pmatrix} 0 & 1 & 1 & 0 \\ f & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -f & 0 \end{pmatrix}$ and $M = h \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ f & 0 & 0 & -1 \\ 0 & -f & -1 & 0 \end{pmatrix}$ where $f \neq 1$; $P = Q = (-1,0)$. Algebra $\mathbb{M}$ is Σ-M-selfdual.

Algebra $\mathbb{N}$: $\Sigma = \begin{pmatrix} 0 & -g & 0 & f \\ g & 0 & f & 0 \\ 0 & f & 0 & -g \\ f & 0 & g & 0 \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 0 & -g & f \\ 0 & 0 & f & -g \\ g & f & 0 & 0 \\ f & g & 0 & 0 \end{pmatrix}$ where $f^2 \neq g^2$; $P = Q = (-1,0)$.

Algebra $\mathbb{O}$: $\Sigma = h \begin{pmatrix} 1 & 0 & 0 & f \\ 0 & -1 & 1 & 0 \\ 0 & f & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $M = h \begin{pmatrix} 1 & 0 & 0 & f \\ 0 & -1 & f & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ where $f \neq 1$; $P = Q = (-1,0)$. A special case is when $f = 0$. The algebra $\mathbb{O}$ is Σ-M-selfdual.

Algebra $\mathbb{P}$: $\Sigma = h\begin{pmatrix} 0 & 0 & 1 & f \\ 0 & 0 & 1 & 1 \\ 1 & -f & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & 0 & f \\ 1 & 0 & -f & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$ where $f \neq 1$; $P = Q = (-1, 0)$. A special case is when $f = 0$. The algebra $\mathbb{P}$ is Σ-M-dual to the algebra $\mathbb{N}$.

Algebra $\mathbb{Q}$: $\Sigma = h\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$; $P = Q = (-1, 0)$.

Algebra $\mathbb{R}$: $\Sigma = M = h\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix}$; $P = Q = (-1, 0)$. So the algebra $\mathbb{R}$ is Σ-M-selfdual.

Algebra $\mathbb{S}$: $\Sigma = M = h\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$; $P = Q = (-1, 0)$. The algebra $\mathbb{S}$ is Σ-M-selfdual.

Algebra $\mathbb{T}$: $\Sigma = h\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$; $P = Q = (-1, 0)$.

Algebra $\mathbb{U}$: $\Sigma = h\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$; $P = Q = (-1, 0)$. The algebras $\mathbb{T}$ and $\mathbb{U}$ are Σ-M-dual.

Subcase 4.3.4. $Q = (1, 0)$. We can make linear transformations of X to make that Σ_{12} is one of the standard forms: $\Sigma_{12} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ or $\Sigma_{12} = \begin{pmatrix} c & 0 \\ 1 & c \end{pmatrix}$. In particular we may assume either $a_{1212} = 0 = a_{1221}$ or $a_{1212} = 0$ and $a_{1221} = 1$.

Let's consider the first case by assuming $a_{1212} = 0 = a_{1221}$. If $a_{1211} = 0 = a_{1222}$, then $\Sigma_{12} = 0$, we don't need to consider this. Otherwise after exchanging x_1 and x_2, we may always assume that $a_{1211} \neq 0$. Replacing the y_i by scalar multiples, we may assume that $a_{1211} = 1$. In addition to those equivalent to the algebras $\mathbb{K}$ and $\mathbb{L}$, System C has two more solutions:

Algebra $\mathbb{V}$: $\Sigma = h\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$; $P = (-1, 0)$ and $Q = (1, 0)$.

Algebra $\mathbb{W}$: $\Sigma = h\begin{pmatrix} 0 & f & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & f \\ 0 & -1 & 1 & 0 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & f & 0 \\ 1 & 0 & 0 & f \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$ where $f \neq -1$; $P = (-1, 0)$ and $Q = (1, 0)$.

In the rest of Subcase 4.3.4, we assume that $a_{1212} = 0$ and $a_{1221} = 1$ and $a_{1211} = a_{1222}$.

System C has two solutions:

Algebra $\mathbb{X}$: $\Sigma = h\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ and $M = h\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$; $P = (-1, 0)$ and $Q = (1, 0)$.

Algebra $\mathbb{Y}$: $\Sigma = h\begin{pmatrix} 1 & 0 & 0 & 0 \\ f & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & f & -1 \end{pmatrix}$ and $M = h\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f & 1 & -1 & 0 \\ 1 & f & 0 & -1 \end{pmatrix}$; $P = (-1, 0)$ and $Q = (1, 0)$.

4.4. *Case four:* $P = (1, 0)$

This is the last piece of the classification. As before we consider the following four subcases.

Subcase 4.4.1. $Q = (1, 1)$. Up to linear transformation all C-solutions give rise to iterated Ore extensions.

Subcase 4.4.2. $Q = (q, 0)$ where $q \neq \pm 1$. Up to linear transformation all C-solutions give rise to iterated Ore extensions.

Subcase 4.4.3. $Q = (1, 0)$. All C-solutions give rise to iterated Ore extensions up to linear transformation.

Subcase 4.4.4. $Q = (-1, 0)$. Up to linear transformation we have two C-solutions which could lead to non-trivial double extensions. The first one is $\Sigma = h\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ and $M = h\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$; $P = (1, 0)$ and $Q = (-1, 0)$. This is a special case of the algebra $\mathbb{D}$. The final case is

Algebra $\mathbb{Z}$: $\Sigma = h\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & f & -1 & 0 \\ f & 0 & 0 & -1 \end{pmatrix}$ and $M = h\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & f & 0 \\ 0 & 1 & 1 & 0 \\ f & 0 & 0 & -1 \end{pmatrix}$ where $f(1 + f) \neq 0$; $P = (1, 0)$ and $Q = (-1, 0)$.

When $f = -1$, the matrix Σ is singular. When $f = 0$, then $\Sigma_{21} = 0$. Note that the algebra $\mathbb{Z}$ is Σ-M-dual to the algebra $\mathbb{W}$. To see this we need to use linear transformations. The M-matrix of the algebra $\mathbb{Z}$ can be changed to

$$M = h\begin{pmatrix} 0 & 1 & \sqrt{f} & 0 \\ 1 & 0 & 0 & -\sqrt{f} \\ \sqrt{f} & 0 & 0 & 1 \\ 0 & -\sqrt{f} & 1 & 0 \end{pmatrix}$$

after a linear transformation $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} \sqrt{f} & 1 \\ \sqrt{f} & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ which is equivalent to the Σ-matrix of the algebra $\mathbb{W}$ up to a linear transformation. This also shows that it is not obvious when two algebras are Σ-M-dual in general.

Finally we list all relations of the algebra $\mathbb{Z}$:

$$\begin{aligned} x_2x_1 &= -x_1x_2, \\ y_2y_1 &= y_1y_2, \\ y_1x_1 &= x_1y_2 + x_2y_2, \\ y_1x_2 &= x_2y_1 + x_1y_2, \\ y_2x_1 &= fx_2y_1 - x_1y_2, \\ y_2x_2 &= fx_1y_2 - x_2y_2. \end{aligned}$$

We summarize what we did in this section in the following proposition that is also part (b) of Theorem 0.1. We use $\mathcal{LIST}$ to denote the class consisting of all these 26 algebras from $\mathbb{A}$ to $\mathbb{Z}$.

Proposition 4.4. *Suppose that $B := (k_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$ is a connected graded double extension with x_1, x_2, y_1, y_2 in degree 1. If B is not an iterated Ore extension of $k_Q[x_1, x_2]$, then the trimmed double extension $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ is isomorphic to one of the algebras in the $\mathcal{LIST}$.*

Remark 4.5. The algebras $\mathbb{E}$ and $\mathbb{J}$ are Σ-M-dual. Consequently, $\mathbb{E}$ and $\mathbb{J}$ are isomorphic to each other by sending x_i's to y_i's. However non-trimmed double extensions extended from the algebras $\mathbb{E}$ and $\mathbb{J}$ may not be isomorphic, because the roles played by x_i's and y_i's are different in the non-trimmed double extensions $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$. For the purpose of finding all non-trimmed double extensions (which is another interesting project), we want to distinguish $\mathbb{E}$ from $\mathbb{J}$ in our list. This remark applies to all Σ-M-dual pairs.

5. Properties of double extensions

In this section we prove that all trimmed double extensions in the $\mathcal{LIST}$ classified in the last section are strongly noetherian, Auslander regular and Cohen–Macaulay. It seems to us that there is no uniform method that works for all algebras, so we have to show this case by case. First we recall a result of [28].

Theorem 5.1. *(See [28, Theorem 0.2].) Let A be a regular algebra. Then any double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ is regular. As a consequence, a double extension of the form $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma, \delta, \tau]$ is regular.*

This theorem ensures the Artin–Schelter regularity for algebras in the $\mathcal{LIST}$. For non-regular rings, it is convenient to use dualizing complexes which were introduced in [24]. We will only use some facts about rings with dualizing complexes and refer to [25] for definitions, properties and other details.

An algebra A is called *strongly noetherian* if for every commutative noetherian (ungraded) ring S, $A \otimes S$ is noetherian [2, p. 580]. Let A be a noetherian algebra with a dualizing complex R. For any left A-module M, the grade of M is defined to be

$$j_R(M) = \inf\{i \mid \operatorname{Ext}^i_A(M, R) \neq 0\}.$$

The grade of a right A-module is defined similarly. The dualizing complex R is called *Cohen–Macaulay* if there is a finite integer d such that

$$j_R(M) + \operatorname{GKdim} M = d$$

for all finitely generated left and right non-zero A-modules M. The dualizing complex R is called *Auslander* [25, Definition 2.1] if the following conditions hold:

(a) for every finitely generated left A-module M, every integer q, and any right A-submodule $N \subset \operatorname{Ext}^q_A(M, R)$, one has $j(N) \geqslant q$;
(b) the same holds after exchanging left with right.

Since A is connected graded, the balanced dualizing complex over A (which is unique) is defined [24]. For simplicity, we say A is *Auslander* (respectively, *Cohen–Macaulay*) if (a) the balanced dualizing complex over A exists and (b) the balanced dualizing complex over A has the Auslander property (respectively, Cohen–Macaulay property).

A noetherian algebra A is called *Auslander regular* if A has finite global dimension and the A-bimodule A as a dualizing complex over A is Auslander. This definition of Auslander regularity is equivalent to the definition given in [9]. To save space we will use the following a temporary term: an algebra is called *SNAC* if it is strongly noetherian, Auslander regular and Cohen–Macaulay.

Remark 5.2.

(a) We will only work with connected graded rings and the dualizing complexes will always be balanced dualizing complexes.
(b) If A is Artin–Schelter regular, then the balanced dualizing complex over A has the form of ${}^{\tau}A(-l)[-d]$ for some graded automorphism τ of A. In this case A is Auslander regular if and only if A is Auslander.

One of the useful properties of dualizing complexes is that the Auslander and Cohen–Macaulay properties are defined for non-regular rings. For example, the Auslander and Cohen–Macaulay properties pass from a graded ring to any of its factor rings without worrying about the regularity.

As we have seen, many double extensions are iterated Ore extensions. Those algebras are Auslander regular and Cohen–Macaulay by the following lemma.

Lemma 5.3. *Let $B := A[t;\sigma,\delta]$ be a connected graded Ore extension of a noetherian algebra A.*

(a) *If A is strongly noetherian, then so is B.*
(b) *If A is Auslander and Cohen–Macaulay, so is B.*
(c) *If A is regular of dimension three, then A is SNAC.*
(d) *If A is commutative (or PI), then A is strongly noetherian, Auslander and Cohen–Macaulay.*

Proof. (a) This is [2, Proposition 4.1(b)].

(b) We construct a connected graded noetherian filtration on B by setting the new degree of t to be $\deg t + 1$, so the associated graded ring of B has the property $\operatorname{gr} B \cong A[t;\sigma]$. By [25, Corollary 6.8], we only need to show the assertion for $A[t;\sigma]$. Then it follows from [25, Theorem 5.1].

(c) For the strongly noetherian property, we note that there is a normal element g of degree 3 such that $A/(g)$ is noetherian of GKdim 2. By [2, Theorem 4.24], $A/(g)$ is strongly noetherian, and by [2, Proposition 4.9], A is strongly noetherian. The rest is [9, Corollary 6.2].

(d) See [2, Proposition 4.9(5)] and [25, Corollary 6.9(i)]. □

As seen in the last section, many algebras are Ore extensions of regular algebras of dimension three. By the above lemma, the following proposition is proved.

Proposition 5.4. *Algebras $\mathbb{A}$, $\mathbb{D}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbb{K}$, $\mathbb{L}$, $\mathbb{Q}$, $\mathbb{V}$, $\mathbb{X}$ and $\mathbb{Y}$ are SNAC.*

Proof. In each of these cases, we have either $M_{12} = 0$ or $\{M_{21} = 0, q_{11} = 0\}$. So every algebra can be written as an Ore extension of a regular algebra of dimension three by Proposition 3.6(b), (c). The assertion follows from Lemma 5.3. □

The following is proved in [28].

Lemma 5.5. *(See [28, Proposition 0.5 and Section 4].) The algebra $\mathbb{R}$ is SNAC.*

The following lemma is well known and follows basically from [26, Theorem 1.3(f), (j)].

Lemma 5.6. *Let B be a graded twist of A in the sense of [26]. Then A is SNAC if and only if B is.*

The usefulness of Lemma 5.6 is that if two algebras A and B are twist equivalent in the sense of Definition 3.4, then A is SNAC if and only if B is. In particular, when we prove a double extension is SNAC, we may assume that $h = 1$, which we will do in the rest of this section.

Algebras in Proposition 5.4 and Lemma 5.5 are SNAC. So we will consider the algebras in the rest of $\mathcal{LIST}$. It is clear that B is SNAC if and only if the opposite ring B^{op} is. If two double extensions are Σ-M-dual, then they are opposite to each other up to equivalences (see Proposition 3.8 and Definition 4.3). Therefore we have the following.

Lemma 5.7. *If algebras A and B are Σ-M-dual, then A is SNAC if and only if B is.*

We can pair together the Σ-M-dual algebras: $(\mathbb{E},\mathbb{J})$, $(\mathbb{F},\mathbb{I})$, $(\mathbb{N},\mathbb{P})$, $(\mathbb{T},\mathbb{U})$ and $(\mathbb{W},\mathbb{Z})$. The other algebras are Σ-M-selfdual: $\mathbb{B},\mathbb{C},\mathbb{M},\mathbb{O},\mathbb{S}$. Basically it reduces to ten algebras to work on.

Recall that the term "Auslander" can be used for non-regular algebras. The facts given in the next lemma will be used freely.

Lemma 5.8. *Let A be a connected graded algebra and let t be a homogeneous normal element of A (not necessarily a non-zero-divisor).*

(a) [2, Proposition 4.9] *A is noetherian* (*respectively, strongly noetherian*) *if and only if* $A/(t)$ *is.*
(b) [25, Theorem 5.1] *Suppose A is noetherian. Then A is Auslander* (*respectively, strongly noetherian, Cohen–Macaulay*) *if and only if* $A/(t)$ *is.*

The following lemma is easy to see. The statements about the strongly noetherian property follow from [2, Proposition 4.1] and other from [25, Theorem 4.16 and Proposition 3.9].

Lemma 5.9. *Let A and B be a connected graded algebras.*

(a) *Suppose* $A \subset B$ *and* B_A *and* ${}_AB$ *are finite. If A is strongly noetherian, Auslander and Cohen–Macaulay, then so is B.*
(b) *Suppose B is a factor ring* A/I*. If A is strongly noetherian, Auslander and Cohen–Macaulay, then so is B.*

Let A be a graded ring and n be a positive integer. The nth *Veronese subring* of A is defined to be

$$A^{(n)} = \bigoplus_{i \in \mathbb{Z}} A_{in}.$$

For later discussion we will use the following special case of Lemma 5.9(a).

Lemma 5.10. *Let* $k_Q[x_1, x_2]^{(2)}$ *be the* 2*nd Veronese subring of* $k_Q[x_1, x_2]$*. Then the algebra* $(k_Q[x_1, x_2])_P[y_1, y_2; \sigma]$ *has the noetherian property if and only if the subalgebra* $(k_Q[x_1, x_2]^{(2)})_P[y_1, y_2; \sigma]$ *does. The same holds for strongly noetherian, Auslander, and Cohen–Macaulay properties.*

The algebra $(k_Q[x_1, x_2]^{(2)})_P[y_1, y_2; \sigma]$ is a double extension of $k_Q[x_1, x_2]^{(2)}$ where σ is the restriction of σ on the 2nd Veronese subring. The next lemma is particularly useful for algebras such as $\mathbb{B}$.

Lemma 5.11. *If* $\Sigma_{11} = \Sigma_{22} = 0$*, then the double extension is SNAC.*

Proof. In this case $\sigma_{11} = 0 = \sigma_{22}$. For every $a \in kx_1 + kx_2$ we have

$$y_1 a = \sigma_{12}(a) y_2 \quad \text{and} \quad y_2 a = \sigma_{21}(a) y_1.$$

Hence for every $a, b \in kx_1 + kx_2$,

$$y_1 ab = \sigma_{12}(a)\sigma_{21}(b) y_1 \quad \text{and} \quad y_2 ab = \sigma_{21}(a)\sigma_{12}(b) y_2.$$

Hence y_1 is normal in $R := (k_Q[x_1, x_2])_P^{(2)}[y_1, y_2; \sigma]$ and y_2 is normal in $R/(y_1)$.

It is well known that $(k_Q[x_1, x_2])^{(2)}$ is strongly noetherian, Auslander and Cohen–Macaulay. By Lemma 5.8(a), (b), so is $R = (k_Q[x_1, x_2])_P^{(2)}[y_1, y_2; \sigma]$. The assertion follows from Theorem 5.1 and Lemma 5.10. $\square$

Here is a consequence of Lemma 5.11.

Proposition 5.12. *The algebras* $\mathbb{B}$, $\mathbb{E}$ $\mathbb{J}$, $\mathbb{N}$ *and* $\mathbb{P}$ *are SNAC.*

Proof. Lemma 5.11 is applied directly to the algebra $\mathbb{B}$, $\mathbb{E}$ and $\mathbb{P}$. For the algebras $\mathbb{J}$ and $\mathbb{N}$, we use the Σ-M-dual property (see Lemma 5.7) and then apply Lemma 5.11. $\square$

We will leave the algebra $\mathbb{C}$ to the end and work on other algebras first.

Proposition 5.13. *The algebras $\mathbb{F}$ and $\mathbb{I}$ are SNAC.*

Proof. Since the algebra $\mathbb{F}$ is Σ-M-dual to the algebra $\mathbb{I}$, we only consider $\mathbb{I}$.

The relations of the algebra $\mathbb{I}$ are

$$\begin{aligned}
x_2x_1 &= qx_1x_2,\\
y_2y_1 &= -y_1y_2,\\
y_1x_1 &= -qx_1y_1 - qx_2y_1 + x_1y_2 - qx_2y_2,\\
y_1x_2 &= x_1y_1 + x_2y_1 + x_1y_2 - qx_2y_2,\\
y_2x_1 &= x_1y_1 + qx_2y_1 + qx_1y_2 - qx_2y_2,\\
y_2x_2 &= -x_1y_1 - qx_2y_1 + x_1y_2 - x_2y_2
\end{aligned}$$

where $q^2 = -1$. We have assumed $h = 1$ by using Lemma 5.6. Using these relations we obtain the following relations

$$\begin{aligned}
y_1(x_1 - x_2) &= (-1-q)(x_1 + x_2)y_1,\\
y_1(x_1 + qx_2) &= (1+q)(x_1 - qx_2)y_2,\\
y_2(x_1 + x_2) &= (1+q)(x_1 - x_2)y_2,\\
y_2(x_1 - qx_2) &= (1+q)(x_1 + qx_2)y_2.
\end{aligned}$$

Using the first and the third relations above we have

$$\begin{aligned}
y_1y_2(x_1 - x_2) &= -y_2y_1(x_1 - x_2) = (1+q)y_2(x_1 + x_2)y_1\\
&= (1+q)^2(x_1 - x_2)y_2y_1 = -(1+q)^2(x_1 - x_2)y_1y_2.
\end{aligned}$$

A similar computation shows that

$$y_1y_2(x_1 + x_2) = -(1+q)^2(x_1 + x_2)y_1y_2.$$

Hence y_1y_2 skew-commutes with x_1 and x_2 with the scalar $-(1+q)^2$, so it is a normal element in the algebra $\mathbb{I}$.

Using the original relations one also sees that

$$\begin{aligned}
(y_1 + qy_2)x_1 &= (-1-q)x_2(y_1 + qy_2),\\
(y_1 - qy_2)x_2 &= (1+q)x_1(y_1 - qy_2).
\end{aligned}$$

Then we have

$$(y_1 + qy_2)(y_1 - qy_2)x_2 = (1+q)(y_1 + qy_2)x_1(y_1 - qy_2) = -(1+q)^2x_2(y_1 + qy_2)(y_1 - qy_2)$$

or

$$\left(y_1^2 + y^2 - 2qy_1y_2\right)x_2 = -(1+q)^2x_2\left(y_1^2 + y_2^2 - 2qy_1y_2\right).$$

Using the relation

$$y_1y_2x_2 = -(1+q)^2x_2y_1y_2$$

which was proved in the last paragraph, we obtain

$$\big(y_1^2+y_2^2\big)x_2 = -(1+q)^2x_2\big(y_1^2+y_2^2\big).$$

By symmetry,

$$\big(y_1^2+y_2^2\big)x_1 = -(1+q)^2x_1\big(y_1^2+y_2^2\big).$$

Hence both y_1y_2 and $y_1^2+y_2^2$ are normal elements in the algebra $\mathbb{I}$. The factor ring $\mathbb{I}/(y_1y_2, y_1^2+y_2^2)$ is a finite module over $k_Q[x_1,x_2]$. The assertion follows from Lemmas 5.8, 5.9 and Theorem 5.1. $\square$

Let a, b, c be elements in an algebra D. We say *a skew-commutes with $kb+kc$* if $a(kb+kc) = (kb+kc)a$ holds in D.

Proposition 5.14. *The algebras $\mathbb{M}$ and $\mathbb{O}$ are SNAC.*

We will only sketch the proof since it is similar to the proof of Proposition 5.13. The general idea is to use Lemmas 5.8, 5.9 and 5.10 properly.

Proof of Proposition 5.14. First we consider the algebra $\mathbb{O}$. By using the four mixing relations of the algebra $\mathbb{O}$, one can show that the element $x_1^2 - fx_2^2$ is normal in the algebra $\mathbb{O}$, where $f \neq 1$ is a parameter appeared in the matrix Σ. Let D' be the algebra $k_Q[x_1,x_2]_P^{(2)}[y_1,y_2;\sigma]$. By Lemma 5.10 we only need to show that D' has the desired properties, namely, D' is strongly noetherian, Auslander and Cohen–Macaulay. By Lemma 5.8(b) we only need to show that the factor ring $D := D'/(x_1^2 - fx_2^2)$ has the desired properties. Now let us introduce some new variables: $X_1 = x_1^2$ and $X_2 = x_1x_2$. Then we have $X_1X_2 = X_2X_1$, $y_1y_2 = -y_2y_1$; and, after identifying fx_2^2 with X_1 in the algebra D, the mixing relations of the algebra $\mathbb{O}$ imply

$$\begin{aligned}
y_1X_1 &= (1+f)X_1y_1 + 2fX_2y_2,\\
y_1X_2 &= (-1-f)X_2y_1 + 2X_1y_2,\\
y_2X_1 &= -2fX_2y_1 + (1+f)X_1y_2,\\
y_2X_2 &= -2X_1y_1 + (-1-f)X_2y_2.
\end{aligned}$$

Use these relations we see that

$$(y_1+iy_2)X_1 = \big((1+f)X_1 - 2ifX_2\big)(y_1+iy_2)$$

where $i^2 = -1$, and

$$(y_1+iy_2)X_2 = \big(-2iX_1 - (1+f)X_2\big)(y_1+iy_2).$$

So y_1+iy_2 skew-commutes with kX_1+kX_2. Similarly, y_1-iy_2 skew-commutes with kX_1+kX_2. Thus all of $(y_1-iy_2)^2$, $(y_1+iy_2)(y_1-iy_2)$ and $(y_1-iy_2)(y_1+iy_2)$ skew-commute with kX_1+kX_2. Note that

$$\{a:=(y_1-iy_2)^2, b:=(y_1+iy_2)(y_1-iy_2), c:=(y_1-iy_2)(y_1+iy_2)\} \tag{E5.14.1}$$

form a k-linear basis for $k_P[y_1,y_2]_2$. Since $y_1y_2=-y_2y_1$, $(y_1-iy_2)^2=(y_1+iy_2)^2$. Let C be the subalgebra of D generated by X_1, X_2, a, b and c (see (E5.14.1)). Then the elements a, b and c are normal in C and the factor ring $C/(a,b,c)$ is generated by X_1 and X_2. Since $X_1X_2=X_2X_1$, Lemma 5.3(d) implies that $C/(a,b,c)$ has the desired properties. By Lemma 5.8, the algebra C has the desired properties. Since D is a finite module over C, D has the desired properties. This finishes the case $\mathbb{O}$.

The proof for case $\mathbb{M}$ is similar. The details are omitted since we can use the proof in the case of the algebra $\mathbb{O}$. Let us only give a few key points. We work inside the ring $D':=k_Q[x_1,x_2]_P^{(2)}[y_1,y_2]$. First we show that $fx_1^2-x_2^2$ is normal. Modulo $fx_1^2-x_2$ in D', the elements y_1+iy_2 and y_1-iy_2 skew-commute with kX_1+kX_2 where $X_1:=x_2^2$ and $X_2=x_1x_2$. Therefore the algebra $\mathbb{M}$ is SNAC. $\square$

Near the end of the proof of Proposition 5.14, we use the fact that $y_1\pm iy_2$ skew-commute with kX_1+kX_2. A similar idea will be used again for the algebra $\mathbb{S}$.

Lemma 5.15. *Let B be an algebra $(k_Q[x_1,x_2])_P[y_1,y_2;\sigma]$ with $Q=(-1,0)$. If x_1-x_2 and x_1+x_2 skew-commute with ky_1+ky_2, then B is SNAC.*

Proof. Let

$$a_1=(x_1+x_2)^2=x_1^2+x_2^2,$$
$$a_2=(x_1-x_2)(x_1+x_2)=x_1^2+2x_1x_2-x_2^2,$$
$$a_3=(x_1+x_2)(x_1-x_2)=x_1^2-2x_1x_2-x_2^2.$$

Hence $\{a_1,a_2,a_3\}$ is a k-linear basis of $k_Q[x_1,x_2]_2$. By the hypotheses, x_1-x_2 and x_1+x_2 skew-commute with ky_1+ky_2. Hence a_1,a_2,a_3 are normal elements in $D:=(k_Q[x_1,x_2]^{(2)})_P[y_1,y_2;\sigma]$. Clearly, the factor ring $D/(a_1,a_2,a_3)$ is isomorphic to $k_P[y_1,y_2]$, which is strongly noetherian, Auslander and Cohen–Macaulay. The assertion follows from Lemmas 5.8, 5.9 and 5.10. $\square$

Proposition 5.16. *The algebras $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$, $\mathbb{W}$ and $\mathbb{Z}$ are SNAC.*

Proof. First we consider the algebra $\mathbb{T}$. The four mixing relations of the algebra $\mathbb{T}$ are

$$y_1x_1=-x_1y_1+x_2y_1+x_1y_2+x_2y_2,$$
$$y_1x_2=x_1y_1-x_2y_1+x_1y_2+x_2y_2,$$
$$y_2x_1=x_1y_1+x_2y_1+x_1y_2-x_2y_2,$$
$$y_2x_2=x_1y_1+x_2y_1-x_1y_2+x_2y_2.$$

Using these we obtain that

$$y_1(x_1+x_2)=2(x_1+x_2)y_2 \quad \text{and} \quad y_2(x_1+x_2)=2(x_1+x_2)y_1.$$

Hence x_1+x_2 skew-commutes with ky_1+ky_2. Similarly, x_1-x_2 skew-commutes ky_1+ky_2. The assertion for the algebra $\mathbb{T}$ follows from Lemma 5.15.

Since the algebra $\mathbb{U}$ is Σ-M-dual to the algebra $\mathbb{T}$, the assertion for the algebra $\mathbb{U}$ follows from Lemma 5.7.

The proof for the algebra $\mathbb{S}$ is very similar to the proof for the algebra $\mathbb{T}$, so the details are omitted.

For the algebra $\mathbb{Z}$, we use Lemma 5.15 again. Note that $Q = (-1, 0)$. The mixing relations of this algebra imply that

$$\begin{aligned} y_1(x_1+x_2) &= (x_1+x_2)(y_1+y_2), \\ y_2(x_1+x_2) &= (x_1+x_2)(fy_1-y_2), \\ y_1(x_1-x_2) &= (x_1-x_2)(y_1-y_2), \\ y_2(x_1-x_2) &= (x_1-x_2)(-fy_1-y_2). \end{aligned}$$

These relations show that $x_1 - x_2$ and $x_1 + x_2$ skew-commute with $ky_1 + ky_2$. The assertion follows from Lemma 5.15.

Since the algebra $\mathbb{W}$ is Σ-M-dual to the algebra $\mathbb{Z}$, the assertion for $\mathbb{W}$ follows from Lemma 5.7. $\square$

The last case to deal with is the algebra $\mathbb{C}$, which is slightly more complicated. The general idea of the proof is the same, but we need to work with elements of degree three (instead of degree two).

Proposition 5.17. *The algebra $\mathbb{C}$ is SNAC.*

Proof. First we list the relations of the algebra $\mathbb{C}$ as follows:

$$y_2y_1 = py_1y_2 \quad \text{and} \quad x_2x_1 = px_1x_2$$

where $p^2 + p + 1 = 0$ (or $p^3 = 1$ and $p \neq 1$); and

$$\begin{aligned} y_1x_1 &= -x_1y_1 + p^2x_2y_1 + x_1y_2 - px_2y_2, \\ y_1x_2 &= -px_1y_1 + x_2y_1 + x_1y_2 - px_2y_2, \\ y_2x_1 &= -px_1y_1 - 2p^2x_2y_1 + px_1y_2 - px_2y_2, \\ y_2x_2 &= -px_1y_1 + p^2x_2y_1 + x_1y_2 - x_2y_2. \end{aligned}$$

Using these relations we obtain the following

$$\begin{aligned} y_1(x_1-x_2) &= (p-1)\big(x_1-p^2x_2\big)y_1, \\ y_1\big(x_1-p^2x_2\big) &= \big(1-p^2\big)(x_1-px_2)y_2, \\ y_2(x_1-px_2) &= \big(p^2-p\big)(x_1-x_2)y_1. \end{aligned}$$

These three relations are used in the following computations:

$$\begin{aligned} y_1^2y_2(x_1-px_2) &= \big(p^2-p\big)y_1^2(x_1-x_2)y_1 \\ &= \big(p^2-p\big)(p-1)y_1\big(x_1-p^2x_2\big)y_1^2 \\ &= \big(p^2-p\big)(p-1)\big(1-p^2\big)(x_1-px_2)y_2y_1^2 \\ &= (p-1)^2\big(1-p^2\big)(x_1-px_2)y_1^2y_2. \end{aligned}$$

The same three relations also imply

$$y_2 y_1^2(x_1 - x_2) = (p-1)^2\big(1-p^2\big)(x_1 - x_2)y_2 y_1^2.$$

Since $y_2 y_1^2 = p^2 y_1^2 y_2$, $y_1^2 y_2$ skew-commutes with $kx_1 + kx_2$ with scalar $(p-1)^2(1-p^2)$. Since $y_1^2 y_2$ skew-commutes with $ky_1 + ky_2$, it is a normal element in the algebra $\mathbb{C}$.

Next we will find another normal element in degree 3. Using the four mixing relations we obtain three other relations:

$$\begin{aligned}
(y_1 - y_2)x_2 &= \big(1-p^2\big)x_2(y_1 - py_2),\\
(y_1 - py_2)x_2 &= \big(p^2 - p\big)x_1\big(y_1 - p^2 y_2\big),\\
\big(y_1 - p^2 y_2\big)x_1 &= (p-1)x_2(y_1 - y_2).
\end{aligned}$$

The first relation of those three shows that

$$(y_1 - y_2)^3 x_2 = \big(1-p^2\big)^3 x_2 (y_1 - py_2)^3.$$

It is easy to show that

$$(y_1 - y_2)^3 = (y_1 - py_2)^3 = \big(y_1 - p^2 y_2\big)^3 = y_1^3 - y_2^3.$$

Thus $y_1^3 - y_2^3$ skew-commutes with x_2 with scalar $(1-p^2)^3$. Using all three relations we obtain that

$$\begin{aligned}
&(y_1 - py_2)(y_1 - y_2)\big(y_1 - p^2 y_2\big)x_1\\
&\quad = (p-1)(y_1 - py_2)(y_1 - y_2)x_2(y_1 - y_2)\\
&\quad = (p-1)\big(1-p^2\big)(y_1 - py_2)x_2(y_1 - py_2)(y_1 - y_2)\\
&\quad = (p-1)\big(1-p^2\big)\big(p^2 - p\big)x_1\big(y_1 - p^2 y_2\big)(y_1 - py_2)(y_1 - y_2).
\end{aligned}$$

An easy computation shows that

$$(y_1 - py_2)(y_1 - y_2)\big(y_1 - p^2 y_2\big) = \big(y_1 - p^2 y_2\big)(y_1 - py_2)(y_1 - y_2) = y_1^3 - y_2^3.$$

Therefore $y_1^3 - y_2^3$ skew-commutes with x_1. Since $y_1^3 - y_2^3$ commutes with both y_1 and y_2, we conclude that $y_1^3 - y_2^3$ is a normal element in the algebra $\mathbb{C}$. After factoring out both elements $y_1^2 y_2$ and $y_1^3 - y_2^3$ in $\mathbb{C}$, the factor ring is a finite module over $k_Q[x_1, x_2]$. By Lemmas 5.8 and 5.9, the algebra $\mathbb{C}$ is SNAC. □

Combining these propositions we have

Theorem 5.18. *The algebras $\mathbb{A}$ to $\mathbb{Z}$ are SNAC.*

We are (almost) ready to prove Theorem 0.1. We refer to [25, Section 6] for the definitions related to noetherian filtrations.

Lemma 5.19. *Let A be a filtered algebra such that the associated graded ring* gr A *is connected graded and SNAC. Then A is SNAC.*

Proof. By [2, Proposition 4.10], A is strongly noetherian. The rest follows from [25, Corollary 6.8]. □

Proof of Theorem 0.1. The regularity follows from Theorem 5.1.

(c) This is Proposition 4.4.

(a) Let B be a double extension $A_P[y_1, y_2; \sigma, \delta, \tau]$ where $A = k_Q[x_1, x_2]$. If B is an iterated Ore extension of A, then the assertion follows from Lemma 5.3. Now we assume B is not an iterated Ore extension of A. By [28, Lemma 3.4], B has a filtration such that $\operatorname{gr} B$ is the trimmed double extension. By Lemma 5.19, it suffices to show that the trimmed Ore extension is SNAC. By part (c), since B is not an iterated Ore extension of A, the trimmed double extension is one of the algebras $\mathbb{A}$ to $\mathbb{Z}$. Therefore the assertion follows from Theorem 5.18.

(b) This is [10, Proposition 1.4]. □

Recall that a regular algebra B is called a normal extension if there is a non-zero-divisor x of degree 1 such that $B/(x)$ is Artin–Schelter regular. Many of the algebras in the $\mathcal{LIST}$ are not isomorphic to either Ore extensions or normal extensions of regular algebras of dimension three. This can be proved by using the method in the proof of [28, Lemmas 4.9 and 4.10].

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